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## FURTHER RIGID TRIPLES OF CLASSES IN $G_2$

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**ABSTRACT.** We establish the existence of two rigid triples of conjugacy classes in the algebraic group  $G_2$  in characteristic 5, complementing results of the second author with Liebeck and Marion. As a corollary, the finite groups  $G_2(5^n)$  are not  $(2, 4, 5)$ -generated, confirming a conjecture of Marion in this case.

### 1. Introduction

Let  $G$  be a connected simple algebraic group over an algebraically closed field  $K$ , and let  $C_1, \dots, C_s$  be conjugacy classes of  $G$ . Following [16], we say the  $s$ -tuple  $\mathbf{C} = (C_1, \dots, C_s)$  is *rigid* in  $G$  if the set

$$\mathbf{C}_0 \stackrel{\text{def}}{=} \{(x_1, \dots, x_s) \in C_1 \times \dots \times C_s : x_1 x_2 \dots x_s = 1\}$$

is non-empty and forms a single orbit under the action of  $G$  by simultaneous conjugation.

Some well-known examples of rigid tuples of classes in simple algebraic groups are the Belyi triples and Thompson tuples, defined in [18]. Other rigid triples are known, see for instance [2, 3, 5, 9, 17]. Rigid tuples of classes are interesting in the context of the inverse Galois problem [12], and also arise naturally in the theory of ordinary differential equations [7].

Recall that a group is  $(a, b, c)$ -generated if it is generated by elements  $x, y$  and  $z$ , of respective orders  $a, b$  and  $c$ , such that  $xyz = 1$ . The group is then called an  $(a, b, c)$ -group, and the triple  $(x, y, z)$  is called an  $(a, b, c)$ -triple of the group. The theory of  $(a, b, c)$ -generation of finite groups has close

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connections to rigidity, for instance it is a basic observation that given a rigid tuple  $\mathbf{C}$  of classes of  $G$ , all subgroups  $\langle x_1, \dots, x_s \rangle$  for  $(x_1, \dots, x_s) \in \mathbf{C}_0$  are conjugate in  $G$ , so that there is at most one  $r > 0$  such that the finite subgroup  $G(p^r)$  is generated by elements in such an  $s$ -tuple.

Let  $K = \overline{\mathbb{F}}_5$  be the algebraic closure of the field of five elements. In [9] it is shown that the simple algebraic group  $G = G_2(K)$  has a rigid triple of conjugacy classes of elements of orders 2, 5 and 5, and any triple of elements  $(x_1, x_2, x_3)$  in the corresponding set  $\mathbf{C}_0$  generates a copy of  $\text{Alt}_5$ . This is then used to show that none of the groups  $G_2(5^n)$ ,  $SL_3(5^n)$  or  $SU_3(5^n)$  is a  $(2, 5, 5)$ -group.

Here we produce two further rigid triples of classes in  $G = G_2(K)$ , closely related to the triple above. Recall from [1] that  $G$  has a unique class of involutions, with representative  $t$ , say, and  $C_G(t) = A_1 \tilde{A}_1$  is a central product of two subgroups  $SL_2(K)$ , where  $A_1$  (resp.  $\tilde{A}_1$ ) is generated by a long (resp. short) root subgroup of  $G$ . There also exist two classes of elements of order 4, with representatives  $s_1$  and  $s_2$ , such that  $C_G(s_1) = A_1 T'$  and  $C_G(s_2) = \tilde{A}_1 T''$ , where  $T'$  and  $T''$  are 1-dimensional tori. Finally, recall from [8] that  $G$  has three classes of unipotent elements of order 5: the long and short root elements, and the class labelled  $G_2(a_1)$ , with representative  $u = x_\beta(1)x_{3\alpha+\beta}(1)$ , where  $\alpha$  (resp.  $\beta$ ) is the short (resp. long) simple root of  $G$ . From [11, Table 22.1.5], the centraliser  $C_G(u) = U_4 \cdot \text{Sym}_3$ , where  $U_4$  is a 4-dimensional connected unipotent group.

### Theorem 1.1.

- (i) *The triples of classes  $\mathbf{C} = (t^G, s_1^G, u^G)$  and  $\mathbf{D} = (t^G, s_2^G, u^G)$  are rigid in  $G = G_2(K)$ .*
- (ii) *Every triple of elements in  $\mathbf{C}_0$  or  $\mathbf{D}_0$  generates a subgroup isomorphic to the symmetric group  $\text{Sym}_5$ .*
- (iii) *None of the groups  $G_2(5^n)$  are a  $(2, 4, 5)$ -group for any  $n$ . Neither are the groups  $SL_3(5^n)$  or  $SU_3(5^n)$ .*

### Remark 1.2.

- (1) *Each subgroup  $\text{Sym}_5$  in part (ii) here contains a subgroup  $\text{Alt}_5$  arising from [9, Theorem 1(ii)].*
- (2) *Keeping track of details in the proof in [9] shows that  $G_2(K)$  has a unique class of subgroups  $\text{Alt}_5$ . These subgroups have centraliser  $\text{Sym}_3$ , and by Lang's theorem these split into three classes in  $G_2(5^r)$ , with centraliser orders 6, 3 and 2. Similarly, if  $S$  and  $S'$  are representatives of the two subgroup classes in part (ii) here, then  $C_G(S) \cong \text{Sym}_3$ , while  $C_G(S')$  is cyclic of order 2. It follows that the class of  $S$  (resp.  $S'$ ) splits into 3 (resp. 2) classes of subgroups in  $G_2(5^r)$ , with centralisers of order 6, 3, 2 (resp. 2 and 2).*
- (3) *A conjecture of Marion [14] states that, for a simple algebraic group  $G$  in characteristic  $p$ , if  $\delta_i$  denotes the dimension of the variety of elements of  $G$  of order  $i$  and if  $\delta_a + \delta_b + \delta_c = 2 \dim(G)$ , then at most finitely many of the finite groups  $G(p^r)$  are  $(a, b, c)$ -groups. For  $G$  of type  $G_2$ , this criterion holds precisely when  $(a, b, c) = (2, 4, 5)$  or  $(2, 5, 5)$ . Hence part (iii), together with [9, Theorem 1(iii)], verifies the conjecture for  $G = G_2$  in characteristic 5. A non-constructive proof of this fact is given in [6, Proposition 3.7(i)], where it is shown that every  $(2, 4, 5)$ -subgroup and*

(2, 5, 5)-subgroup of  $G_2(K)$  is reducible on the natural 7-dimensional module, by considering the dimensions of  $SL_7(K)$ -conjugacy classes of elements in the relevant  $(a, b, c)$ -triples.

### 2. Proof of the Theorem

We proceed in the manner of [9]. Let  $G = G_2(K)$  and  $t, u, s_1, s_2 \in G$  as above. If  $\sigma$  is a Frobenius morphism of  $G$  induced from the field map  $x \mapsto x^5$  of  $K$ , then

$$G = \bigcup_{n=1}^{\infty} G_{\sigma^n} = \bigcup_{n=1}^{\infty} G_2(5^n).$$

The element  $u = x_{\beta}(1)x_{3\alpha+\beta}(1)$  is a regular unipotent element in a subgroup  $A_2 = SL_3(K)$  of  $G$  generated by long root groups, and therefore lies in a subgroup  $\Omega_3(5) \cong \text{Alt}_5$  of  $G$ , which we denote by  $A$ . Now, let  $S = N_{A_2}(A) = SO_3(5) \cong \text{Sym}_5$ . Following the proof given in [9] we find that  $N_G(A) = S \times C_G(A)$  and  $C_G(A) = \langle z, \tau \rangle \cong \text{Sym}_3$ , where  $\langle z \rangle$  is the centre of  $A_2$  and  $\tau$  is an outer involution in  $N_G(A_2) = A_2.2$ . Note that  $C_{A_2}(\tau) = SO_3(K)$ , so  $\tau \in C_G(S)$ .

Let  $v$  be an involution in  $S \setminus A$ , so that  $S = \langle A, v \rangle$ , and define  $S' = \langle A, v\tau \rangle$ , so that  $S' \cong \text{Sym}_5$  also. Then  $C_G(S), C_G(S') \leq C_G(A) = \langle z, \tau \rangle$  and therefore

$$(2.1) \quad C_G(S) = \langle z, \tau \rangle,$$

$$(2.2) \quad C_G(S') = \langle \tau \rangle.$$

In particular  $S$  and  $S'$  are not conjugate in  $G$ .

Next consider the set of  $(2, 4, 5)$ -triples of  $\text{Sym}_5$ . It is straightforward to show that there are exactly 120 such triples, and that  $\text{Sym}_5$  acts transitively on these by simultaneous conjugation.

Now let  $\mathbf{C} = (t^G, s_1^G, u^G)$  and  $\mathbf{D} = (t^G, s_2^G, u^G)$ , and for  $q$  a fixed power of 5 let  $\mathbf{C}_0(q) = \mathbf{C}_0 \cap G_2(q)^3$  and  $\mathbf{D}_0(q) = \mathbf{D}_0 \cap G_2(q)^3$ . We now show that  $|\mathbf{C}_0(q)| = |\mathbf{D}_0(q)| = |G_2(q)|$ . For this we require the character table of  $G_2(q)$ , given in [1] and available in the CHEVIE [4] computational package. Since  $C_G(u)/C_G(u)^\circ = S_3$ , an application of Lang’s theorem [13, Theorem 21.11] shows that  $u^G \cap G_2(q)$  splits into three classes, with representatives denoted in [1] by  $u_3, u_4$  and  $u_5$ , and respective centraliser orders  $6q^4, 3q^4$  and  $2q^4$ . For  $x, y, z \in G_2(q)$  let  $a_{xyz}$  be the corresponding class algebra constant. Calculations with the character table show that

$$a_{ts_i u_j} = \begin{cases} q^4 & \text{if } i = 1, j \in \{3, 4, 5\} \text{ or } i = 2, j = 4, \\ 3q^4 & \text{if } i = 2, j = 3, \\ 0 & \text{if } i = 2, j = 5. \end{cases}$$

and it follows that

$$|\mathbf{C}_0(q)| = \sum_{j=3}^5 |u_j^{G_2(q)}| a_{ts_1 u_j} = |G_2(q)| \left( \frac{q^4}{6q^4} + \frac{q^4}{3q^4} + \frac{q^4}{2q^4} \right) = |G_2(q)|,$$

$$|\mathbf{D}_0(q)| = \sum_{j=3}^5 |u_j^{G_2(q)}| a_{ts_2 u_j} = |G_2(q)| \left( \frac{3q^4}{6q^4} + \frac{q^4}{2q^4} \right) = |G_2(q)|.$$

Now let  $\mathbf{E}$  denote (resp.  $\mathbf{E}'$ ) denote the set of triples  $(x_1, x_2, x_3) \in \mathbf{C}_0 \cup \mathbf{D}_0$  which generate a conjugate of  $S$  (resp. a conjugate of  $S'$ ). Then  $G$  is transitive on both  $\mathbf{E}$  and  $\mathbf{E}'$ , since if  $\langle x_1, x_2, x_3 \rangle = \langle y_1, y_2, y_3 \rangle^g$  are each isomorphic to  $\text{Sym}_5$ , then  $(x_1^g, x_2^g, x_3^g)$  and  $(y_1, y_2, y_3)$  are  $(2, 4, 5)$  triples in a fixed copy of  $\text{Sym}_5$ , hence conjugate in  $\text{Sym}_5$  by the observation above. Moreover both  $\mathbf{E}$  and  $\mathbf{E}'$  are non-empty, since  $S$  and  $S'$  each contain  $(2, 4, 5)$ -triples and a unique conjugacy class of unipotent elements, whose elements are conjugate to an element of  $A$  and therefore are conjugate to  $u$ . By (2.1) and (2.2) the stabiliser of a point in  $\mathbf{E}$  is isomorphic to  $\text{Sym}_3$ , and the stabiliser of a point in  $\mathbf{E}'$  is cyclic of order 2. Hence applying Lang's theorem shows that  $\mathbf{E}(q) = \mathbf{E} \cap G_2(q)^3$  splits into three  $G_2(q)$ -orbits, of orders  $|G_2(q)|/r$  for  $r = 2, 3, 6$ , and similarly  $\mathbf{E}'(q) = \mathbf{E}' \cap G_2(q)^3$  splits into two orbits, each of order  $|G_2(q)|/2$ . Therefore,

$$|\mathbf{E}(q)| + |\mathbf{E}'(q)| = |G_2(q)| \left( \frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = |\mathbf{C}_0(q)| + |\mathbf{D}_0(q)|$$

and it follows that  $\mathbf{C}_0(q) \cup \mathbf{D}_0(q) = \mathbf{E}(q) \cup \mathbf{E}'(q)$  for each  $q$ . Therefore

$$\mathbf{C}_0 \cup \mathbf{D}_0 = \bigcup_{n=1}^{\infty} \mathbf{C}_0(5^n) \cup \mathbf{D}_0(5^n) = \bigcup_{n=1}^{\infty} \mathbf{E}(5^n) \cup \mathbf{E}'(5^n) = \mathbf{E} \cup \mathbf{E}'$$

Hence  $G$  has exactly two orbits on  $\mathbf{C}_0 \cup \mathbf{D}_0$ . A triple in  $\mathbf{C}_0$  cannot lie in the same orbit as a triple in  $\mathbf{D}_0$  since the corresponding elements of order 4 are not  $G$ -conjugate, and it follows that the two  $G$ -orbits are  $\mathbf{C}_0$  and  $\mathbf{D}_0$ .

This proves parts (i) and (ii) of the Theorem. For part (iii), suppose that  $G_2(5^n)$ ,  $SL_3(5^n)$  or  $SU_3(5^n)$  is a  $(2, 4, 5)$ -group, with corresponding set of generators  $x_1, x_2, x_3$ . Since  $L(G_2) \downarrow A_2$  is a direct sum of  $L(A_2)$  and two 3-dimensional irreducible  $A_2$ -modules (cf. [10, Table 8.5]), it follows that  $C_{L(G_2)}(x_1, x_2, x_3) = 0$ . An application of a result of Scott [15] to the module  $L(G)$ , as in the proof of [16, Corollary 3.2], then yields

$$\dim(x_1^G) + \dim(x_2^G) + \dim(x_3^G) \geq 2 \dim(G) = 28,$$

implying  $(x_1^G, x_2^G, x_3^G) = \mathbf{C}$  or  $\mathbf{D}$ , which contradicts part (ii) of the Theorem.  $\square$

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