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REFINEMENTS OF THE BELL AND STIRLING NUMBERS

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Abstract. We introduce new refinements of the Bell, factorial, and unsigned Stirling numbers of the first and second kind that unite the derangement, involution, associated factorial, associated Bell, incomplete Stirling, restricted factorial, restricted Bell, and r-derangement numbers (and probably more!). By combining methods from analytic combinatorics, umbral calculus, and probability theory, we derive several recurrence relations and closed form expressions for these numbers. By specializing our results to the classical case, we recover explicit formulae for the Bell and Stirling numbers as sums over compositions.

1. **Introduction**

The Bell and Stirling numbers have been studied for over a century because of their importance to many combinatorial problems. They frequently arise in enumeration problems in combinatorics, and also satisfy many complex identities and inter-relations [4, 11]. In this paper, we present refinements of the Bell and Stirling numbers that preserve many essential structural properties of their classical counterparts. The associated generating functions have a simple form which enables us to follow classical proofs of many identities. Generalizing to an arbitrary index set S also allows us to unite several previously disparate generalizations of the Stirling and Bell numbers.

We use a variety of methods to explore their properties; we begin with an analytic combinatoric construction of their generating function, followed by umbral and generating function methods to

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derive identities for these numbers. Finally, an explicit probabilistic representation allows us to construct some new and extremely interesting expressions for the Bell and Stirling numbers as sums over compositions.

Let S be a nonempty (and possibly infinite) set of indices. We consider **S-Bell numbers** $B_{n,S}$, defined as the number of ways to partition n into blocks, where each block has a size which is found in S. We also consider the **S-factorial numbers** $A_{n,S}$, which count the number of permutations of n labeled elements into cycles, such that each cycle has a number of elements that belongs to the index set S. For example, with $S = \{1, 3\}$, $B_{4,S} = 5$ and $A_{4,S} = 9$. The relevant partitions for $B_{4,S}$ are

- {{1},{2},{3},{4}},
- {{1,2,3},{4}},
- {{1,2,4},{3}},
- {{1,3,4},{2}},
- {{2,3,4},{1}},

and the relevant permutations for $A_{4,S}$ are

- (1)(2)(3)(4)
- (1)(342)
- (1)(432)
- (2)(341)
- (2)(431)
- (3)(241)
- (3)(421)
- (4)(231)
- (4)(321).

We introduce

$$\begin{bmatrix} n \\ k \end{bmatrix}_S,$$

the **(unsigned) S-Stirling numbers of the first kind**, which count the number of permutations of n elements with k cycles, such that each cycle has a cardinality found in S. We also introduce

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_S,$$

the **S-Stirling numbers of the second kind**, which count the number of partitions of n into k blocks, such that each block has a cardinality found in S. These have two free variables, and contain more combinatorial informations than the Bell and factorial numbers.

We also need to define initial values, since setting the following values allows us to concisely state consistent generating function identities. These definitions are consistent with initial values for the classical case [4]. For $n \geq 1$, we set

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_S = \begin{bmatrix} 0 \\ k \end{bmatrix}_S = \begin{Bmatrix} n \\ 0 \end{Bmatrix}_S = \begin{Bmatrix} 0 \\ k \end{Bmatrix}_S = 0.$$

We also set

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_S = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}_S = 1,$$

and note that from the combinatorial definition

$$\begin{bmatrix} n \\ k \end{bmatrix}_S = \begin{Bmatrix} n \\ k \end{Bmatrix}_S = 0$$

for $k > n$.

Directly from the combinatorial definitions, we see that we have the equations

$$B_{n,S} = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_S$$

and

$$A_{n,S} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_S .$$

Therefore, we set the initial values

$$A_{0,S} = B_{0,S} = 1.$$

These numbers have been studied for many special values of the index set S. The case $S = \mathbb{Z}_{\geq 1} = \{1, 2, \ldots\}$ yields the classical Bell, Stirling, and factorial numbers since these do not have any restrictions on cycle length or block size. The cases $S = \{1, 2, \ldots, k\}$ and $S = \{k, k+1, \ldots\}$ are referred to as $B_{n,\leq k}$, $B_{n,\geq k}$, $A_{n,\leq k}$, and $A_{n,\geq k}$, and have been extensively studied several times [12, 13]. In another direction, $S = \{1, 2\}$ means that $B_{n,S} = A_{n,S}$ yield the involution numbers [1] and $S = \{2, 3, 4, \ldots\}$ leads to the derangement numbers [3]. Extending our study of Stirling numbers from these particular index sets S to the general case allows us to prove general theorems that apply to every special case. It also leads to identities that are very non-intuitive combinatorially, guiding future work.

Table 1. Specializations

Set	**Classical object**
$\{1, 2, 3, \ldots\}$	Bell and Stirling, $n!$
$\{k, k+1, \ldots\}$	associated Bell, associated Stirling, associated factorial (r-derangement)
$\{1, 2, 3, \ldots, k\}$	restricted Bell, restricted Stirling, restricted factorial
$\{1, 2\}$	involution numbers
$\{2, 3, 4, \ldots\}$	derangement numbers

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2. Symbolic Methods

Here we present the basic ideas of *analytic combinatorics*, which trivializes the proof of our generating functions. For further details of the theory described in this section, refer to Flajolet and Sedgewick [7]. The main idea of the symbolic method is to write our combinatorial constructions as the composition of several basic operations on a single element. We can then algorithmically read off a generating function for our combinatorial quantity from this sequence of operations.

We consider exponential generating functions (EGFs), which naturally correspond to labeled objects. Some of the most basic constructions *for labeled elements* (there are subtle differences for unlabeled objects) are SUM, $PROD$, SEQ, SET, and CYC. Given that $A(z) = \sum_n a_n \frac{z^n}{n!}$ and $B(z) = \sum_n b_n \frac{z^n}{n!}$ are the EGFs of $\{a_n\}$ and $\{b_n\}$, the operation SUM produces an EGF for $\{a_n + b_n\}$ - the number of elements of size n, following pointwise addition of A and B. This is trivially $A(z) + B(z) = \sum_n (a_n + b_n) \frac{z^n}{n!}$, but other operations allow us to construct very nontrivial generating functions.

The remaining operations are:

- $PROD(\mathcal{B}, \mathcal{C})$, which gives an EGF for the cardinalities of the "labeled product" of $\{a_n\}$ and $\{b_n\}$, which consists of the set of ordered pairs $\{(a_n, b_n)\}$ after an order consistent relabeling. This is given by $A(z) \times B(z)$.
- $SEQ(\mathcal{B})$, which corresponds to an EGF for the number of sequences of a given size with parts in $\mathcal{B}$, is given by $\frac{1}{1-B(z)}$. $SEQ_k(\mathcal{B})$, corresponding to the size of all k-element sequences, is given by $B(z)^k$.
- $SET(\mathcal{B})$ corresponds to forming all sequences, taken modulo an equivalence relation identifying all sequences that are permutations of each other. This is given by $\exp(B(z))$. $SET_k(\mathcal{B})$, the set of all k-sequences modulo this same equivalence relation, is given by $\frac{B(z)^k}{k!}$.
- $CYC(\mathcal{B})$ corresponds to $SEQ(\mathcal{B})$, taken modulo an equivalence relation identifying sequences whose elements are *cyclic* permutations of each other, and is given by $\log \frac{1}{1-B(z)}$. $CYC_k(\mathcal{B})$, the set of k-sequences modulo this same equivalence relation, is given by $\frac{B(z)^k}{k}$.

Using just these few operations (there are more!) allows us to write

Theorem 2.1. *We have the following generating functions:*

$$\sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_S \frac{z^n}{n!} = \frac{1}{k!} \left(\sum_{s \in S} \frac{z^s}{s} \right)^k,$$

$$\sum_{n=0}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix}_S \frac{z^n}{n!} = \frac{1}{k!} \left(\sum_{s \in S} \frac{z^s}{s!} \right)^k,$$

$$\sum_{n=0}^{\infty} A_{n,S} \frac{z^n}{n!} = \exp\left(\sum_{s \in S} \frac{z^s}{s} \right),$$

$$\sum_{n=0}^{\infty} B_{n,S} \frac{z^n}{n!} = \exp\left(\sum_{s \in S} \frac{z^s}{s!} \right).$$

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Proof. We give an example for how to translate a combinatorial definition into a symbolic construction, from which we can easily read off the corresponding generating function. In what follows, we let $\mathcal{Z}$ represent an "atomic class" of a single element of size one; the given EGFs follow from interpreting $\mathcal{Z}$ as the variable z, and formally applying the rules for symbolic constructions.

We begin with the combinatorial definition of $\begin{bmatrix} n \\ k \end{bmatrix}_S$, the number of permutations of n labeled elements with k cycles, such that each cycle has a cardinality found in S. We first form the sum $\sum_{s\in S} CYC_s(\mathcal{Z})$. Since each CYC_s operator is invariant under a cyclic permutation of its sequences, it enumerates the number of cycles of length s. The sum over $s \in S$ corresponds to the fact that we allow a cycle to have any cardinality found in our index S. SET_k then composes k cycles, while ignoring the order in which the cycles are permuted. The coefficient of z^n in the corresponding EGF will therefore count the number of ways n elements can be decomposed as permutations with cycle lengths in S – which is precisely $\begin{bmatrix} n \\ k \end{bmatrix}_S$. Therefore, we have the equivalence

$$\sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_S \frac{z^n}{n!} = SET_k\left(\sum_{s\in S} CYC_s(\mathcal{Z})\right).$$

The analysis for $\begin{Bmatrix} n \\ k \end{Bmatrix}_S$ roughly follows the one before. However, since the S-Stirling numbers of the second kind count the number of partitions of n into k *blocks* instead of k *cycles*, we apply inner SET operators instead of CYC operators, corresponding to the symbolic construction

$$\sum_{n=0}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix}_S \frac{z^n}{n!} = SET_k\left(\sum_{s\in S} SET_s(\mathcal{Z})\right).$$

When considering the EGF of $B_{n,S}$ we repeat the analysis for $\begin{bmatrix} n \\ k \end{bmatrix}_S$, but now we apply an outer SET operator instead of SET_k, since the number of cycles is arbitrary instead of being equal to k. The construction for $A_{n,S}$ follows similarly, giving us the dual identities

$$\sum_{n=0}^{\infty} A_{n,S}\frac{z^n}{n!} = SET\left(\sum_{s\in S} CYC_s(\mathcal{Z})\right),$$
$$\sum_{n=0}^{\infty} B_{n,S}\frac{z^n}{n!} = SET\left(\sum_{s\in S} SET_s(\mathcal{Z})\right).$$

Translating each of these constructions into EGFs completes the proof. □

When we take $S = \{1, 2, 3, \ldots\}$, we note that $\sum_{s\in S} \frac{z^s}{s!} = e^x - 1$ and we recover the EGF for the classical Bell numbers e^{e^x-1}. In addition, by taking $S = \{1, 2, \ldots, m\}$ and $\{m, m+1, m+2, \ldots\}$, we recover the generating functions $\exp\left(\sum_{i=1}^m \frac{z^i}{i!}\right)$, $\exp\left(e^x - \sum_{i=0}^m \frac{z^i}{i!}\right)$, and so on [13, Thm. 4.2, Thm. 5.2, (4.3)].

The last two generating functions show that $B_{n,S}$ and $A_{n,S}$ are special cases of the *complete Bell polynomials*, which are defined by the generating function

$$\exp\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\right) = \sum_{n=0}^{\infty} B_n(a_0, \ldots, a_n)\frac{t^n}{n!}.$$

By comparing generating functions, we recover $B_{n,S}$ through the choice

$$a_n = \begin{cases} 1, & n \in S; \\ 0, & n \notin S. \end{cases}$$

Analogously, we recover $A_{n,S}$ through the choice

$$a_n = \begin{cases} (n-1)!, & n \in S; \\ 0, & n \notin S. \end{cases}$$

3. Composition sums

Since all of the generating functions in Theorem 2.1 can be easily represented as the composition of two functions, we can apply the **Faà di Bruno formula** for higher order derivatives to derive new formulae for $A_{n,S}$, $B_{n,S}$, and the S-Stirling numbers.

Throughout this paper, we will consider **compositions** of an integer n. A composition π is an *ordered* tuple of positive integers called **parts** that add up to n – therefore, $(1,1,2)$, $(1,2,1)$, and $(2,1,1)$ all represent different compositions of 4. We denote by $\mathcal{C}$ the set of all compositions, by $\mathcal{C}_n$ the set of all compositions of n, and by $|\cdot|$ the number of parts in a composition. For example, $|(1,1,2)| = 3$. For the composition $\pi \in \mathcal{C}_n$, we denote the set of its parts by $\{\pi_i\}$. We also use the multi-index notation $g_\pi := \prod_{\pi_i \in \pi} g_{\pi_i}$ and $\pi! = \prod_{\pi_i \in \pi} \pi_i!$, which enables us to state a convenient corollary of the classical Faà di Bruno formula:

Theorem 3.1. [20, Thm. 9] *Let $g(z) = \sum_{n\geq 1} g_n z^n$ and $f(z) = \sum_{n\geq 0} f_n z^n$. We then have the generating function identity*

$$f(g(z)) = f_0 + \sum_{n\geq 1} z^n \sum_{\pi \in \mathcal{C}_n} f_{|\pi|} g_\pi. \tag{3.1}$$

Theorem 3.2. *We have the following composition sum identities:*

$$\begin{bmatrix} n \\ k \end{bmatrix}_S \frac{1}{n!} = \sum_{\substack{\pi \in \mathcal{C}_n \\ \pi_i \in S \\ |\pi| = k}} \frac{1}{|\pi|! \prod_{\pi_i \in \pi} \pi_i},$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_S \frac{1}{n!} = \sum_{\substack{\pi \in \mathcal{C}_n \\ \pi_i \in S \\ |\pi| = k}} \frac{1}{|\pi|! \pi!},$$

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$$\frac{A_{n,S}}{n!} = \sum_{\substack{\pi\in\mathcal{C}_n\\ \pi_i\in S}} \frac{1}{|\pi|!\prod_{\pi_i\in\pi}\pi_i},$$

$$\frac{B_{n,S}}{n!} = \sum_{\substack{\pi\in\mathcal{C}_n\\ \pi_i\in S}} \frac{1}{|\pi|!\pi!}.$$

Proof. To prove the formula for $A_{n,S}$, apply Theorem 3.1 with $f(z) = e^z$ and $g(z) = \sum_{s\in S}\frac{z^s}{s}$. Therefore,

$$g_\pi = \begin{cases} \frac{1}{\prod_{\pi_i\in\pi}\pi_i}, & \text{if every part of } \pi \text{ is} \in S; \\ 0, & \text{otherwise.} \end{cases}$$

This allows us to rewrite our composition sum as follows:

$$\sum_{\pi\in\mathcal{C}_n} f_{|\pi|}g_\pi = \sum_{\pi\in\mathcal{C}_n} \frac{1}{|\pi|!\prod_{\pi_i\in\pi}\pi_i}\mathbb{1}_{\pi_i\in S} = \sum_{\substack{\pi\in\mathcal{C}_n\\ \pi_i\in S}} \frac{1}{|\pi|!\prod_{\pi_i\in\pi}\pi_i},$$

where the restriction on the parts of S is transferred to the summation. Comparing coefficients completes the proof. The formula for $B_{n,S}$ is proven identically with $g(z) = \sum_{s\in S}\frac{z^s}{s!}$ instead.

To prove the formula for $\begin{bmatrix} n \\ k \end{bmatrix}_S$, apply Theorem 3.1 with $f(z) = \frac{z^k}{k!}$ and $g(z) = \sum_{s\in S}\frac{z^s}{s}$. Now we have the piecewise definition

$$f_{|\pi|} = \begin{cases} \frac{1}{|\pi|!}, & |\pi| = k \\ 0; & \text{otherwise.} \end{cases}$$

We can then simplify the composition sum since

$$\sum_{\pi\in\mathcal{C}_n} f_{|\pi|}g_\pi = \sum_{\substack{\pi\in\mathcal{C}_n\\ \pi_i\in S}} \frac{1}{|\pi|!\prod_{\pi_i\in\pi}\pi_i}\mathbb{1}_{|\pi|=k} = \sum_{\substack{\pi\in\mathcal{C}_n\\ \pi_i\in S\\ |\pi|=k}} \frac{1}{|\pi|!\prod_{\pi_i\in\pi}\pi_i}.$$

The analysis for $\begin{Bmatrix} n \\ k \end{Bmatrix}_S$ proceeds identically, but with $g(z) = \sum_{s\in S}\frac{z^s}{s!}$ instead. □

In the case $S = \mathbb{Z}_{\geq 1}$, we recover known expressions for the classic Stirling and Bell numbers. We can simply leave the restriction $\pi_i \in S$ off the summation, since we're now allowing all positive parts.

An avenue for further research is to find a combinatorial proof of any of the above identities. The appearance of a composition sum is not coincidental; it has a very natural connection to all of these numbers. Take the S-Bell numbers as an example: we care about how we can partition n elements into blocks, where each block has a size found in S. *This is precisely an unordered composition of n with each part in S.* When passing from compositions to partitions we are overcounting in some way which is accounted for by the composition sum.

Corollary 3.3. *We have the inequalities*

$$\begin{bmatrix} n \\ k \end{bmatrix}_S \geq \begin{Bmatrix} n \\ k \end{Bmatrix}_S$$

and

$$A_{n,S} \geq B_{n,S}.$$

Proof. For $\pi \in \mathcal{C}_n$, we have $\pi! \geq \prod_{\pi_i \in \pi} \pi_i$ with equality if and only if $\pi_i = 1$ – that is, $\pi = (1, 1, \ldots, 1)$. Looking at the composition sum expressions in Theorem 3.2, this inequality will therefore be satisfied termwise. □

4. Recurrences

We can now derive several general recurrences satisfied by these numbers, the first two of which generalize [13, Thm. 4.2, Thm. 5.2]. All of the following identities also have combinatorial and umbral proofs; however, the generating function approach appears to be the most concise for the following simple identities. We also note that since all four types considered in this work are specializations of the Bell polynomials, several of the recurrences described here are specializations of recurrences for the Bell polynomials.

Theorem 4.1. *We have the recurrences*

$$A_{n+1,S} = \sum_{s \in S} \frac{n!}{(n-s+1)!} A_{n-s+1,S},$$

$$B_{n+1,S} = \sum_{s \in S} \binom{n}{s-1} B_{n-s+1,S},$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_S = \sum_{s \in S} \frac{n!}{(n-j+1)!} \begin{bmatrix} n-s+1 \\ k-1 \end{bmatrix}_S,$$

$$\begin{Bmatrix} n+1 \\ k \end{Bmatrix}_S = \sum_{s \in S} \binom{n}{s-1} \begin{Bmatrix} n-s+1 \\ k-1 \end{Bmatrix}_S.$$

Proof. For concreteness we work only with the S-Bell numbers. Taking a derivative of the generating function from Theorem 2.1, $\sum_{n=0}^{\infty} B_{n,S} \frac{z^n}{n!} = \exp\left(\sum_{s \in S} \frac{z^s}{s!}\right)$, gives

$$\sum_{n=0}^{\infty} B_{n+1,S} \frac{z^n}{n!} = \left(\sum_{s \in S} \frac{z^{s-1}}{(s-1)!}\right) \exp\left(\sum_{s \in S} \frac{z^s}{s!}\right) = \left(\sum_{s \in S} \frac{z^{s-1}}{(s-1)!}\right) \left(\sum_{n=0}^{\infty} B_{n,S} \frac{z^n}{n!}\right).$$

Taking a Cauchy product and comparing coefficients completes the proof. The other proofs follow exactly the same lines, and follow from taking derivatives and comparing coefficients. □

We also present a combinatorial proof of the second recurrence; the rest follow from similar reasoning. Start with $n+1$ elements and distinguish the first one. We can partition these $n+1$ elements in $B_{n+1,S}$ ways. However, we can also add $s-1$ elements to the first one, for any cardinality index $s \in S$. There are $\binom{n}{s-1}$ ways to pick these $s-1$ elements, and then $B_{n-(s-1),S}$ ways to partition the rest. Summing over s completes the proof.

We can also derive *lacunary recurrences*, which are recurrences for $B_{n,S\cup\sigma}$ in terms of only a subset of the terms $\{B_{1,S}, B_{2,S}, \ldots, B_{n-1,S}\}$. In particular, the formulae below have σ completely free so they sum over arbitrarily few terms if σ is large enough relative to n. We also note that these recover

[13, Thm 4.1, Thm. 4.6] in the case $S = \{1, 2, \ldots, m-1\}$ (or $S = \{1, 2, \ldots, m\}$ for Theorem 4.6) and $\sigma = m$, where it was derived combinatorially.

Theorem 4.2. *If $\sigma \notin S$, we have the recurrence*

$$B_{n,S\cup\sigma} = \sum_{i=0}^{\lfloor \frac{n}{\sigma} \rfloor} \frac{n!}{i!(n-\sigma i)!(\sigma!)^i} B_{n-\sigma i,S}.$$

If $\sigma \in S$, we have

$$B_{n,S\setminus\sigma} = \sum_{i=0}^{\lfloor \frac{n}{\sigma} \rfloor} \frac{n!(-1)^i}{i!(n-\sigma i)!(\sigma!)^i} B_{n-\sigma i,S}.$$

Proof. We begin with the generating function from Theorem 2.1, $\sum_{n=0}^{\infty} B_{n,S} \frac{z^n}{n!} = \exp\left(\sum_{s\in S} \frac{z^s}{s!}\right)$.

Assuming that $\sigma \notin S$, we then multiply both sides by $\exp\left(\frac{z^\sigma}{\sigma!}\right)$. On one side, this gives

$$\exp\left(\frac{z^\sigma}{\sigma!}\right) \exp\left(\sum_{s\in S} \frac{z^s}{s!}\right) = \exp\left(\sum_{s\in S\cup\sigma} \frac{z^s}{s!}\right) = \sum_{n=0}^{\infty} B_{n,S\cup\sigma} \frac{z^n}{n!}.$$

On the other side, this gives

$$\exp\left(\frac{z^\sigma}{\sigma!}\right) \sum_{n=0}^{\infty} B_{n,S} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{\sigma n}}{(\sigma!)^n n!} \times \sum_{n=0}^{\infty} B_{n,S} \frac{z^n}{n!} = \sum_{n=0}^{\infty} z^n \sum_{i=0}^{\lfloor \frac{n}{\sigma} \rfloor} \frac{1}{i!(n-\sigma i)!(\sigma!)^i} B_{n-\sigma i,S}.$$

Comparing coefficients completes the proof. The second formula follows in an identical manner, after multiplying by $\exp\left(-\frac{z^\sigma}{\sigma!}\right)$ instead. □

We can also prove a similar theorem about $A_{n,S}$ using exactly the same reasoning.

Theorem 4.3. *If $\sigma \notin S$, we have the recurrence*

$$A_{n,S\cup\sigma} = \sum_{i=0}^{\lfloor \frac{n}{\sigma} \rfloor} \frac{n!}{i!(n-\sigma i)!\sigma^i} A_{n-\sigma i,S}.$$

If $\sigma \in S$,

$$A_{n,S\setminus\sigma} = \sum_{i=0}^{\lfloor \frac{n}{\sigma} \rfloor} \frac{n!(-1)^i}{i!(n-\sigma i)!\sigma^i} A_{n-\sigma i,S}.$$

Special cases of this theorem are [13, Thm. 5.1, Thm. 5.5]. We can also prove other properties based on splitting S, generalizing [13, Thm. 4.7]. We note that $\binom{n}{n_1, n_2, \ldots}$ is the multinomial coefficient $\frac{n!}{n_1! n_2! \cdots}$, and $\mathcal{C}_n$ is the set of compositions of n, described in the previous section.

Theorem 4.4. *Let $S = \cup_i S_i$, where $S_i \cap S_j = \emptyset$ for $i \neq j$, so that the sets $\{S_i\}$ partition S. Then*

$$B_n = \sum_{\pi\in\mathcal{C}_n} \binom{n}{\pi_1, \pi_2, \ldots} B_{\pi_1,S_1} B_{\pi_2,S_2} \cdots,$$

and

$$A_n = \sum_{\pi\in\mathcal{C}_n} \binom{n}{\pi_1, \pi_2, \ldots} A_{\pi_1,S_1} A_{\pi_2,S_2} \cdots.$$

Proof. We begin with the generating function for $B_{n,S}$, split it into parts corresponding to each S_i, and then re-express this as a convolution. In particular,

$$\sum_{n=0}^{\infty} B_{n,S}\frac{z^n}{n!} = \exp\left(\sum_{s\in S}\frac{z^s}{s!}\right) = \prod_i \exp\left(\sum_{s\in S_i}\frac{z^s}{s!}\right) = \prod_i \sum_{n=0}^{\infty} B_{n,S_1}\frac{z^n}{n!}.$$

Computing coefficients of z^n in the resulting product completes the proof. □

Corollary 4.5. *Let $S = \cup_i S_i$, where $S_i \cap S_j = \emptyset$ for $i \neq j$ so that the sets $\{S_i\}$ partition S, and let p be a prime. Then*

$$B_{p,S} \equiv \sum_i B_{p,S_i} \pmod p,$$

and

$$A_{p,S} \equiv \sum_i A_{p,S_i} \pmod p.$$

Proof. If a composition π has each part strictly smaller than p then $\binom{p}{\pi_1,\pi_2,\ldots} \equiv 0 \pmod p$ since $\binom{p}{\pi_1,\pi_2,\ldots} = \frac{p!}{\pi_1!\pi_2!\cdots}$ has a power of p in the numerator which is not canceled by one in the denominator. Every term in the summation then vanishes modulo p, save for those of the form $\pi = (0, 0, \ldots, \pi_i = p, \ldots, 0)$. For each of these compositions we note the initial values $A_{0,S_j} = B_{0,S_j} = 1$, which completes the proof. □

We now discuss *Spivey's formula* [17],

$$B_{n+m} = \sum_{k=0}^{n}\sum_{j=0}^{m} j^{n-k} \left\{ m \atop j \right\} \binom{n}{k} B_k.$$

The cases $n = 0$ and $m = 1$ are

$$B_m = \sum_{j=0}^{m} \left\{ m \atop j \right\}$$

and

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k,$$

which are the two most basic recurrences satisfied by the Bell numbers. These have the analogs

$$B_{m,S} = \sum_{j=0}^{m} \left\{ m \atop j \right\}_S$$

and (from Theorem 4.1)

$$B_{n+1,S} = \sum_{s\in S} \binom{n}{s-1} B_{n-s+1,S},$$

suggesting the existence of a Spivey-type formula for the S-Bell numbers. However, this analog has been particularly difficult to find.

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5. **Umbral approach**

The goal of *umbral calculus* is to simplify the proof of many identities by turning manipulation of sequences into manipulations of moments; this effectively turns questions about subscripts into questions about superscripts. We can do this if, for a sequence $\{a_n\}$, we can find a measure μ such that

$$a_n = \int x^n d\mu\left(x\right).$$

Ideally, μ would be a probability measure, i.e. a positive measure with unit total integral (which implies $a_0 = 1$), but it is important to note that umbral calculus does not require that μ is a probability measure. However, not every sequence of real numbers $\{a_n\}$ can be a (probabilistic) moment sequence. Basic probabilistic results such as $\text{Var}(\mathcal{A}) = \mathbb{E}\mathcal{A}^2 - (\mathbb{E}\mathcal{A})^2 \geq 0$ impose strong constraints on the sequence $\{a_n\}$. For example, we cannot have the moment sequence $\{a_1 = 1, a_2 = 0\}$ since this yields a negative variance.

However, if we are given a sequence that can be written as the moments of a measure μ, we can then rewrite identities on sequences as identities on the moments of their underlying random variables. For a concrete example, consider the expression

$$\sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k} = \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\mathcal{A}_1^k \mathbb{E}\mathcal{A}_2^{n-k} = \mathbb{E} \sum_{k=0}^{n} \binom{n}{k} \mathcal{A}_1^k \mathcal{A}_2^{n-k} = \mathbb{E}(\mathcal{A}_1 + \mathcal{A}_2)^n, \tag{5.1}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are two independent random variables. Now, based on the moments of the distribution $\mathcal{A}_1 + \mathcal{A}_2$, we obtain a nontrivial identity for the sequence $\{a_n\}$. For example, take $\mathcal{A}_1 \sim \Gamma_{p_1}$ and $\mathcal{A}_2 \sim \Gamma_{p_2}$ where $\sim$ denotes equality of distributions. We have let Γ_p denote a gamma-distributed random variable with density

$$\frac{1}{\Gamma\left(p\right)} e^{-x} x^{p-1},$$

so that $\mathcal{A}_1 + \mathcal{A}_2 \sim \Gamma\left(p_1 + p_2\right)$. Since the moments of a Gamma random variable are the rising factorials

$$\mathbb{E}\Gamma_p^k = (p)_k := \prod_{i=0}^{k-1} (p+i),$$

identity (5.1) translates into

$$\sum_{k=0}^{n} \binom{n}{k} \left(p_1\right)_k \left(p_2\right)_{n-k} = \left(p_1 + p_2\right)_n, \tag{5.2}$$

which is nothing but the Chu-Vandermonde identity.

As it can be seen from this example, the key ideas at play here are the linearity of the expectation and the fact that $\mathbb{E}\mathcal{A}_1\mathcal{A}_2 = \mathbb{E}\mathcal{A}_1 \times \mathbb{E}\mathcal{A}_2$ for independent random variables. This allows us to transform questions involving the sequence $\{a_n\}$, which *a priori* has no structure, into questions about moments of a random variable, which behave like powers. For more background for umbral approaches, consult [8, 16].

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After some annoying reverse engineering, we can explicitly construct a random variable which has the complete Bell polynomials as a moment sequence, showing that **the Bell polynomials behave umbrally**. They are in fact special cases of a *Sheffer sequence*, which is one of the most general polynomial sequences which satisfies umbral relations. The construction is quite involved, but allows us to systematize the derivation of identities for the Bell polynomials through umbral methods. In the next section, we will specialize this analysis to obtain results for the S-Bell and S-Stirling numbers.

We first describe a *stable random variable*, which is a random variable X such that there exist constants c_n and d_n so that

$$X_1 + X_2 + \cdots + X_n \sim c_n X + d_n,$$

where the $\{X_i\}$ are independent and identically distributed copies of X. It is a general result [15, Chapter 1] that a variable is stable if and only if it is equal to $aZ + b$, where Z has characteristic function

$$\mathbb{E}\exp(iuZ) = \begin{cases} \exp\left(-|u|^\alpha \left(1 - i\beta \operatorname{sign}(u) \tan\left(\frac{1}{2}\alpha\pi\right)\right)\right), & \alpha \neq 1; \\ \exp\left(-|u| \left(1 + i\beta \operatorname{sign}(u) \frac{2}{\pi} \log(u)\right)\right), & \alpha = 1. \end{cases}$$

This is a two parameter distribution under the restrictions $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$.

We introduce the *symmetric α-stable distribution* S_α, which is the case $\beta = 0$. It is symmetric around the origin and has moment generating function $\mathbb{E}e^{u\mathcal{S}_\alpha} = e^{u^\alpha}$. Note that we still have the very strict requirement $0 < \alpha \leq 2$, which is why we cannot assign $\mathcal{A}_j \sim S_j$ in the results that follow.

Theorem 5.1. *Let $\mathcal{A}_j = \mathcal{W}_j \mathcal{S}_{\frac{1}{j}}^{-\frac{1}{j}}$ be a product of independent random variables. Here,*

$$Pr\left\{\mathcal{W}_j = \exp\left(\frac{2\pi i l}{j}\right)\right\} = \frac{1}{j},$$

$0 \leq l \leq j-1$, so $\mathcal{W}_j$ is a complex-valued random variable equiprobable on the j-th roots of unity. $\mathcal{S}_{1/j}$ is a symmetric α-stable distribution with characteristic parameter $\frac{1}{j}$.

Then the random variable

$$\mathcal{A} \sim \sum_{j=0}^{\infty} a_j^{\frac{1}{j}} (j!)^{-\frac{1}{j}} \mathcal{A}_j,$$

where the $\{\mathcal{A}_j\}$ are independent random variables distributed as above, has moment generating function

$$\mathbb{E}e^{t\mathcal{A}} = \sum_{j=0}^{\infty} B_j(a_0, \ldots, a_n) \frac{t^j}{j!}.$$

Hence,

$$\mathbb{E}\mathcal{A}^n = B_n(a_0, \ldots, a_n).$$

Proof. The variable $\mathcal{A}_j$ has been explicitly described before, and has moments given by [18, 19]

$$\mathbb{E}\mathcal{A}_j^n = \begin{cases} 0, & n \not\equiv 0 \pmod{j}; \\ \frac{(qj)!}{q!}, & n = qj. \end{cases}$$

Therefore, it has moment generating function

$$\mathbb{E}e^{t\mathcal{A}_j} = \sum_{n=0}^{\infty} \mathbb{E}\mathcal{A}_j^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!}\frac{(jn)!}{n!} = e^{t^j}.$$

Due to the independence of the $\{\mathcal{A}_j\}$, we then know that

$$\mathbb{E}e^{t\mathcal{A}} = \prod_{j=0}^{\infty} \mathbb{E}e^{a_j^{\frac{1}{j}}(j!)^{-\frac{1}{j}}t\mathcal{A}_j} = \prod_{j=0}^{\infty} e^{a_j\frac{t^j}{j!}} = \exp\left(\sum_{j=0}^{\infty} a_j \frac{t^j}{j!}\right) = \sum_{j=0}^{\infty} B_j(a_0,\ldots,a_n)\frac{t^j}{j!},$$

which completes the proof. □

We can then exploit this representation to prove umbral identities. Most of the recurrence relations proven earlier also have simple umbral proofs. We can also construct a *conjugate* variable for $\mathcal{A}_j$ – a random variable $\tilde{\mathcal{A}}_j$ that is independent of $\mathcal{A}_j$ and satisfies

$$\mathbb{E}\left(\mathcal{A}_j + \tilde{\mathcal{A}}_j\right)^n = \delta_{n0},$$

which will be extremely useful in generating umbral identities. Here, δ_{ij} is the Kronecker delta function.

Theorem 5.2. *Let $\tilde{\mathcal{A}}_j \sim \exp\left(\frac{\pi i}{j}\right)\mathcal{A}_j$; then $\tilde{\mathcal{A}}_j$ is conjugate to $\mathcal{A}$. Moreover,*

$$\tilde{\mathcal{A}} \sim \sum_{j=0}^{\infty} a_j^{\frac{1}{j}}(j!)^{-\frac{1}{j}}\tilde{\mathcal{A}}_j,$$

where the $\{\tilde{\mathcal{A}}_j\}$ are independent random variables, is conjugate to $\mathcal{A}$. Hence,

$$\mathbb{E}(\mathcal{A}+\tilde{\mathcal{A}})^n = \delta_{n0}.$$

Proof. Two independent random variables $\mathcal{X}$ and $\mathcal{Y}$ are conjugate if and only if $\mathbb{E}e^{t\mathcal{X}}e^{t\mathcal{Y}} = 1$; we can show this by expanding the exponential as a series and appealing to linearity of expectation. Therefore,

$$\mathbb{E}e^{t\mathcal{X}}e^{t\mathcal{Y}} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbb{E}(\mathcal{X}+\mathcal{Y})^n = 1,$$

and $E(\mathcal{X}+\mathcal{Y})^n = \delta_{n0}$.

$\mathcal{A}_j$ and $\tilde{\mathcal{A}}_j$ can be seen as conjugate after manually comparing moment generating functions: $\tilde{\mathcal{A}}_j$ has moments

$$\mathbb{E}\tilde{\mathcal{A}}_j^n = \begin{cases} 0, & n \not\equiv 0 \pmod{j}; \\ \frac{qj!}{q!}(-1)^q, & n = qj, \end{cases}$$

so

$$\mathbb{E}e^{t\tilde{\mathcal{A}}_j} = \sum_{n=0}^{\infty}\mathbb{E}\tilde{\mathcal{A}}_j^n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{t^{jn}}{jn!}\frac{jn!}{n!}(-1)^n = e^{-t^j}.$$

Then, $\mathbb{E}e^{t\mathcal{A}_j}e^{t\tilde{\mathcal{A}}_j} = e^{t^j}e^{-t^j} = 1$ and we are done. Due to the independence of the $\{\tilde{\mathcal{A}}_j\}$, we then compute

$$\mathbb{E}e^{t\tilde{\mathcal{A}}} = \prod_{j=0}^{\infty}\mathbb{E}e^{a_j^{\frac{1}{j}}(j!)^{-\frac{1}{j}}t\tilde{\mathcal{A}}_j} = \prod_{j=0}^{\infty} e^{-a_j\frac{t^j}{j!}} = \exp\left(-\sum_{j=0}^{\infty}a_j\frac{t^j}{j!}\right).$$

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Using this expression, $\mathbb{E}e^{t\mathcal{A}}e^{t\tilde{\mathcal{A}}} = 1$ and we are done. □

While an umbral approach simply requires us to verify that a sequence is a moment sequence, by explicitly constructing the relevant random variable we can obtain a slew of new expressions for the Bell and Stirling numbers. By specializing Theorem 5.1 to the Bell and factorial numbers, we have the following important corollaries:

Corollary 5.3. *Let* $\{\mathcal{A}_j\}$ *be a set of independent random variables as given in Theorem 5.1. Then for*

$$\mathcal{A} \sim \sum_{s\in S} (s!)^{-\frac{1}{s}} \mathcal{A}_s,$$

we have

$$\mathbb{E}\mathcal{A}^n = B_{n,S}.$$

Corollary 5.4. *Let* $\{\mathcal{A}_j\}$ *be a set of independent random variables as given in Theorem 5.1. Then for*

$$\mathcal{A} \sim \sum_{s\in S} s^{-\frac{1}{s}} \mathcal{A}_s,$$

we have

$$\mathbb{E}\mathcal{A}^n = A_{n,S}.$$

6. **Next steps**

These results suggest the study of other types of restricted Bell numbers. An example would be "even" or "odd" Bell numbers, which arise from taking $S_e := \{2, 4, 6, \ldots\}$ and $S_o := \{1, 3, 5, \ldots\}$. The even and odd Bell numbers that count the number of partitions of n into blocks of even (respectively, odd) size. While all of the above theorems apply to the even and odd Bell numbers, we also note that we have such results as $B_n = \sum_{k=0}^{n} \binom{n}{k} B_{k,S_e} B_{n-k,S_o}$. This means that further information about the even and odd Bell numbers could yield new information about the classical case. Another interesting approach would be to compute the valuations of these dissections with respect to small primes, and how that relates to the p-adic valuation of the Bell numbers. We already initiated work in this direction with Corollary 4.5.

In fact, work has already been done on the dissection of B_n, depending on whether it splits n into an even or odd number of blocks – the so called 'complementary Bell numbers'. This is the subject of *Wilf's Conjecture*, which asks whether the two values are ever equal [5]. However, our methods are more suited towards dissecting B_n into whether it splits n into blocks of even or odd size.

In any case, our generating functions provide *refinements* of their classical counterparts, which enables us to dissect the classical case in various ways. For instance, the rank of a partition naturally dissects partitions into residues classes modulo 5. It provides a refinement of the classical partition counting function $p(n)$. Analogously, we could consider Bell and Stirling numbers consisting of block sizes modulo a prime p. When $p = 2$, we are led to the even and odd Bell numbers previously discussed. However, we could consider the sets $S_p^{(1)} = \{1, p+1, 2p+1, \ldots\}$, $S_p^{(2)} = \{2, p+2, p+2, \ldots\}$, and more

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generally $S_p^{(i)} = \{i, p+i, 2p+i, \ldots\}$ for $1 \leq i \leq p$. Then $\mathbb{Z}_{\geq 1} = \cup_{i=1}^{p} S_p^{(i)}$ and we recover the classical case as the multinomial sum

$$B_n = \sum_{\pi \in \mathcal{C}_n} \binom{n}{\pi_1, \pi_2, \ldots} B_{\pi_1, S_p^{(1)}} B_{\pi_2, S_p^{(2)}} \cdots .$$

Moving away from the Bell numbers, there are still many fundamental properties of the S-Stirling numbers that need to be explored. The Stirling numbers satisfy many classical recurrences and orthogonality relations, and it is an interesting question whether the S-Stirling numbers satisfy similar identities. For instance, if $1 \in S$ then matrices related to the S-Stirling numbers have well defined inverses with combinatorial interpretations [6]. In some cases, the answer is no, and it appears that refining the Stirling numbers to an arbitrary index set S destroys some essential structural properties.

For instance, even in the special case $S = \{1, 2, \ldots, m\}$ the restricted Stirling numbers only satisfy the three term recurrences

$$\left\{ {n+1 \atop k} \right\}_S = k \left\{ {n \atop k} \right\}_S + \left\{ {n \atop k-1} \right\}_S - \binom{n}{m} \left\{ {n-m \atop k-1} \right\}_S$$

and

$$\left[{n+1 \atop k} \right]_S = n \left[{n \atop k} \right]_S + \left[{n \atop k-1} \right]_S - \frac{n!}{(n-m)!} \left[{n-m \atop k-1} \right]_S .$$

This appears to be the 'best possible' recurrence, in the sense that it involves a minimal number of terms. We conjecture that this implies that the S-Stirling numbers *cannot be represented as symmetric polynomials*. Following [10, Chapter 1], when considering the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$, the elementary symmetric function $e_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ is defined by the generating product $\prod_{i=1}^{n}(1 + tx_i) = \sum_{k=0}^{\infty} e_k t^k$ and the complete symmetric function $h_k(x_1, \ldots, x_n) = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ is defined by the generating product $\prod_{i=1}^{n}(1 - tx_i)^{-1} = \sum_{k=0}^{\infty} h_k t^k$. These are uniquely characterized by their initial values and the two term recurrences

$$e_{n-j}(x_1, \ldots, x_{n-1}) = e_{n-j}(x_1, \ldots, x_{n-2}) + x_{n-1} e_{n-j-1}(x_1, \ldots, x_{n-2})$$

and

$$h_{n-j}(x_1, \ldots, x_j) = h_{n-j}(x_1, \ldots, x_{j-1}) + x_j h_{n-j-1}(x_1, \ldots, x_j).$$

Since the S-Stirling numbers in general do not satisfy similar two term recurrences, we assume that they are not evaluations of elementary symmetric polynomials. Other generalizations of the Stirling numbers are, however, symmetric polynomial evaluations – [14] uses this technique of matching recurrence relations to establish that the *Jacobi-Stirling numbers* are in fact symmetric polynomial evaluations.

This is important because it means that there is no simple product factorization of the generating function for the Stirling numbers. More concretely, the classical formulae $\prod_{i=0}^{n-1}(x+i) = \sum_{k=0}^{n} \left[{n \atop k} \right] x^k$

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and $x^n = \sum_{k=0}^n \left\{ n \atop k \right\} (x)(x-1)\cdots(x-k+1)$, expressing changes of bases of $\mathbb{Z}[x]$, have no general analog.

Alternatively, Stirling numbers arise as coefficients of the *normal form* expansion of the differential operator $x\frac{d}{dx}$:

$$\left(x\frac{d}{dx}\right)^n = \sum_{k=0}^n \left\{ n \atop k \right\} x^k \frac{d^k}{dx^k}.$$

The work [2] provides a comprehensive overview of studies about normal forms of differential operators. It remains to be seen whether there exists a modified differential operator which contains $\left\{ n \atop k \right\}$ in its normal form expansion.

Another interesting possibility for further work is relating S-Bell numbers to evaluations of classical special functions. Applying Taylor's formula (where $D^n f(x) := \frac{d^n}{dx^n} f(x)|_{x=0}$) to our generating function gives us

$$\frac{B_{n,S}}{n!} = D^n \exp\left(\sum_{s\in S} \frac{z^s}{s!}\right).$$

This yields a connection to the Hermite polynomials, which are defined as proportional to $D^n e^{-x^2}$. When $S = \{1,2\}$ (which classically yields the involution numbers), we can relate special values of $A_{n,S} = B_{n,S}$ and the Hermite polynomials. More generally, Gould-Hopper polynomials are defined in terms of nested derivatives of the exponentials of polynomials. Special values of those could probably be linked to $B_{n,S}$ in the future.

In this paper, we have only established a few basic properties for the S-Bell, S-Stirling, and S-factorial numbers. Their general definitions allow us to unite the study of many previously studied combinatorial quantities, and will hopefully continue to yield new results about the classical Bell and Stirling numbers.

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