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## IOTA ENERGY OF WEIGHTED DIGRAPHS

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ABSTRACT. The eigenvalues of a digraph are the eigenvalues of its adjacency matrix. The iota energy of a digraph is recently defined as the sum of absolute values of imaginary part of its eigenvalues. In this paper, we extend the concept of iota energy of digraphs to weighted digraphs. We compute the iota energy formulae for the positive and negative weight directed cycles. We also characterize the unicyclic weighted digraphs with cycle weight  $r \in [-1, 1] \setminus \{0\}$  having minimum and maximum iota energy. We obtain well known McClelland upper bound for the iota energy of weighted digraphs. Finally, we find the class of noncospectral equienergetic weighted digraphs.

### 1. Introduction

A weighted digraph  $D$  is a pair  $(D^u, \omega)$ , where  $D^u = (V, A)$  is the underlying digraph of  $D$  and  $\omega : A \rightarrow \mathbb{R} \setminus \{0\}$  is the weight function. That is, each arc of  $D$  is assigned a weight from the set of non-zero real numbers. The elements of  $V$  are called vertices and elements of  $A$  are called arcs. If there is an arc from a vertex  $u$  to vertex  $v$ , we denote it by  $uv$ . An arc with a weight is called a weighted arc. The symbol  $w(e)$  is used to denote the weight of any arc  $e = uv$ . The sign of a weighted arc is the sign of its weight. If the direction of weighted arcs are removed from the underlying digraph  $D^u$ , then  $(D^u, \omega)$  is called a weighted graph.

The weight of a weighted digraph  $D$ , denoted by  $w(D)$ , is defined as the product of weights of the arcs of  $D$ . The weighted digraph  $D$  is said to be positive if  $w(D) > 0$  and negative if  $w(D) < 0$ .

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A directed weighted path  $P_n$  of length  $n - 1$  is a weighted digraph on  $n$  vertices  $v_1, v_2, \dots, v_n$  with  $n - 1$  weighted arcs  $v_i v_{i+1}$ ,  $i = 1, 2, \dots, n - 1$ . A directed weighted cycle  $C_n$  of length  $n$ , ( $n \geq 2$ ) is a weighted digraph having vertices  $v_1, v_2, \dots, v_n$  and weighted arcs  $v_i v_{i+1}$ ,  $i = 1, 2, \dots, n - 1$  and  $v_n v_1$ . A weighted digraph is said to be cycle-balanced (respectively, cycle-unbalanced) if each of its cycle has positive weight (respectively, negative weight). The all-positive (respectively, all-negative) weighted digraph  $D^+$  (respectively,  $D^-$ ) of  $D$  is the weighted digraph obtained from  $D$  by replacing weight  $w(e)$  of each arc  $e$  of  $D$  by  $|w(e)|$  (respectively,  $-|w(e)|$ ). A linear weighted digraph is a digraph whose all components are cycles. Throughout this paper, we denote a positive weight cycle of order  $n$  by  $C_n^+$  and a negative weight cycle of order  $n$  by  $C_n^-$ . The sign of a weighted digraph is defined as the product of signs of its arcs. For a detail study on weighted graphs and digraphs see [3, 7, 8].

The adjacency matrix  $A(D) = [a_{ij}]_{n \times n}$  of an  $n$ -vertex weighted digraph  $D$  is defined as

$$a_{ij} = \begin{cases} w(v_i, v_j) & \text{if there is an arc from } v_i \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of a weighted digraph  $D$  is the polynomial  $\phi_D(\lambda) = \det(A(D) - \lambda I_n)$ , where  $I_n$  is the identity matrix of order  $n$ . The eigenvalues of a weighted digraph  $D$  are the eigenvalues of its adjacency matrix  $A(D)$ . The spectrum of  $D$  is the set of eigenvalues of a weighted digraph  $D$  together with their multiplicities. It is denoted by  $\text{spec}(D)$ .

In 1978, Gutman [15] introduced the concept of energy of a simple graph. He defined the energy of a graph as the sum of the absolute values of its eigenvalues. The concept of energy in signed graphs was introduced by Germina et al. [14] in 2010. For more study on energy of signed graphs we refer [6, 5]. The notion of energy to digraphs was extended by Peña and Rada [22] in 2008. Since the adjacency matrix of digraphs is not necessarily symmetric, its eigenvalues may be complex. The energy of a digraph is defined as the sum of the absolute values of real parts of its eigenvalues. Pirzada and Bhat [23] extended the concept of energy to sidigraphs. Khan et al. [19] introduced the notion of iota energy of digraphs as the sum of absolute values of the imaginary parts of its eigenvalues. Farooq et al. [11] extended the concept of iota energy to sidigraphs. The definition of iota energy of a sidigraph is similar to the definition of the iota energy of a digraph. For study on iota energy of bicyclic sidigraphs we refer [10]. The energy of a weighted graph is defined by Gutman and Shao [15, 16] as the sum of absolute values of its eigenvalues. Bhat [4] extended the cocncept of energy to weighted digraphs. The author characterizes the unicyclic weighted digraphs with cycle weight  $r \in [-1, 1] \setminus \{0\}$  having minimum and maximum energy and calculates some bounds for the energy of weighted digraphs.

The following theorem is known as the coefficient theorem for weighted digraphs

**Theorem 1.1.** (Achariya [1]) *If  $D$  is the weighted digraph with characteristic polynomial*

$$\Phi_D(\lambda) = \lambda^n + \sum_{k=1}^n a_k(D) \lambda^{n-k},$$

then

$$a_k(D) = \sum_{L \in \mathcal{L}_k} (-1)^{p(L)} |w(L)| s(L),$$

for all  $k = 1, 2, \dots, n$ , where  $\mathcal{L}_k$  is the set of all linear weighted subdigraphs  $L$  of  $D$  of order  $k$ ,  $p(L)$  denotes the number of components of  $L$ ,  $w(L)$  denotes the weight of linear weighted subdigraph  $L$  and  $s(L)$  denotes the sign of  $L$ .

Two weighted digraphs are said to be cospectral if they have same spectrum, otherwise noncospectral. The following theorem is a spectral characterization of cycle-balanced weighted digraphs [1]

**Theorem 1.2.** (Achariya [1]) *A weighted digraph  $D$  is balanced if and only if it is cospectral with its all positive weighted digraphs  $D^+$ .*

## 2. Energy and Iota energy of weighted digraphs

Motivated by Farooq et al. [11] and Bhat [4] we extend the concept of iota energy to weighted digraphs. Let  $D$  be a weighted digraph with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Then Bhat [4] defined the energy of  $D$  as

$$E(D) = \sum_{k=1}^n |\operatorname{Re}(\lambda_k)|,$$

where  $\operatorname{Re}(\lambda_k)$  denote the real part of the eigenvalue  $\lambda_k$ . Next, we define the iota energy of  $D$  as

$$E_c(D) = \sum_{k=1}^n |\operatorname{Im}(\lambda_k)|,$$

where  $\operatorname{Im}(\lambda_k)$  denote the imaginary part of the eigenvalue  $\lambda_k$ .

**Example 2.1.** *By Theorem 1.1, the characteristic polynomial of an acyclic weighted digraph of order  $n$  is given as  $\phi_D(\lambda) = \lambda^n$ . So its iota energy is  $E_c(D) = 0$ .*

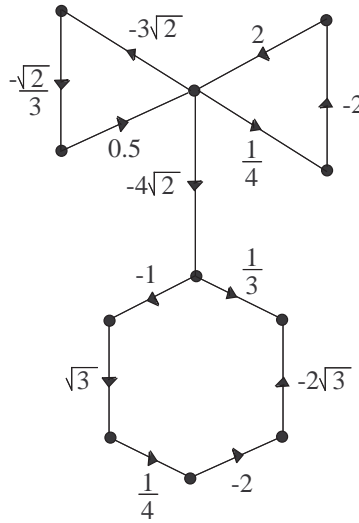


FIGURE 1.

**Example 2.2.** Consider the weighted digraph  $D$  shown in Figure 1. By Theorem 1.1, the characteristic polynomial of  $D$  is given by

$$\phi_D(\lambda) = \lambda^{11} + \lambda^5 = \lambda^5(\lambda^6 + 1)$$

The eigenvalues of  $D$  are  $0^5, \pm i, \frac{-i \pm \sqrt{3}}{2}, \frac{i \pm \sqrt{3}}{2}$ . Thus

$$E_c(D) = 4.$$

A weighted digraph  $D$  is said to be strongly connected if for any two vertices  $u$  and  $v$  of  $D$ , there is a path from  $u$  to  $v$  and a path from  $v$  to  $u$ . The strong components of a weighted digraph are maximally connected weighted subdigraphs. The direct sum of two weighted digraphs  $D_1$  and  $D_2$ , denoted by  $D_1 \oplus D_2$ , is the weighted digraph with  $V(D_1 \oplus D_2) = V(D_1) \cup V(D_2)$  and  $A(D_1 \oplus D_2) = A(D_1) \cup A(D_2)$ .

The next theorem gives the relation between energy of weighted digraph and energy of its strong components.

**Theorem 2.3.** (Bhat [4]) Energy of a weighted digraph is sum of energies of its strong components.

The following theorem is an analogue of the Theorem 2.3.

**Theorem 2.4.** Iota energy of a weighted digraph is sum of iota energies of its strong components.

*Proof.* Proof is similar to the proof of Theorem 2.3 in [4]. □

Next two lemmas will be useful in proving several results.

**Lemma 2.5.** (Khan et al. [18]) Let  $x, a, b$  be real numbers such that  $x \geq a > 0$  and  $b > 0$ . Then we have

$$\frac{\pi x}{b x^2 - \pi^2} \leq \frac{\pi a}{b a^2 - \pi^2}.$$

**Lemma 2.6.** (Farooq et al. [9]) For  $x \in (0, \frac{\pi}{2}]$ , the following inequality holds:

$$\frac{1}{x} - 0.429x \leq \cot x \leq \frac{1}{x} - \frac{x}{3}.$$

For any real number  $x$  with  $0 < x < \frac{\pi}{2}$ , sine function satisfies the following:

$$(2.1) \quad x - \frac{x^3}{3!} \leq \sin x \leq x.$$

**Lemma 2.7.** Consider the sequences  $\langle a_n \rangle$  and  $\langle b_n \rangle$ , where  $a_n$  and  $b_n$  are given by

$$a_n = \begin{cases} 2 \cot \frac{\pi}{n} & \text{if } n \equiv 0 \pmod{2} \\ \cot \frac{\pi}{2n} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

and

$$b_n = \begin{cases} 2 \csc \frac{\pi}{n} & \text{if } n \equiv 0 \pmod{2} \\ \cot \frac{\pi}{2n} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Then  $\{a_n\}$  is strictly increasing for  $n \geq 4$  and  $\{b_n\}$  is strictly increasing for  $n \geq 3$ .

*Proof.* We will show that  $a_n < a_{n+1}$  for  $n \geq 4$ .

Let  $n \equiv 0 \pmod{2}$ . Then  $n + 1 \equiv 1 \pmod{2}$ . By using Lemma 2.6, we get

$$(2.2) \quad \begin{aligned} a_n &= 2 \cot \frac{\pi}{n} \leq 2 \left( \frac{n}{\pi} - \frac{\pi}{3n} \right) \\ &= \frac{2n}{\pi} - \frac{2\pi}{3n}. \end{aligned}$$

On the other hand, for  $n \geq 4$ , we find that

$$\begin{aligned} a_{n+1} &= \cot \frac{\pi}{2(n+1)} \\ &\geq \left( \frac{1}{\frac{\pi}{2(n+1)}} - 0.429 \frac{\pi}{2(n+1)} \right) \\ &= \frac{2n}{\pi} + \frac{2}{\pi} - 0.429 \frac{\pi}{2(n+1)}. \end{aligned}$$

As  $n \geq 4$ , so  $n + 1 \geq 5$ . The above equation becomes

$$(2.3) \quad \begin{aligned} a_{n+1} &\geq \frac{2n}{\pi} + \frac{2}{\pi} - 0.429 \frac{\pi}{10} \\ &\geq \frac{2n}{\pi} + 0.5018. \end{aligned}$$

From (2.2) and (2.3), we get  $a_n < a_{n+1}$ .

Now let  $n \equiv 1 \pmod{2}$ . Then  $n + 1 \equiv 0 \pmod{2}$ . By using Lemma 2.6, we get

$$(2.4) \quad \begin{aligned} a_n &= \cot \frac{\pi}{2n} \\ &\leq \frac{2n}{\pi} - \frac{\pi}{6n}. \end{aligned}$$

On the other hand,

$$\begin{aligned} a_{n+1} &= 2 \cot \frac{\pi}{n+1} \\ &\geq 2 \left( \frac{1}{\frac{\pi}{(n+1)}} - 0.429 \frac{\pi}{n+1} \right) \\ &= \frac{2n}{\pi} + \frac{2}{\pi} - 0.858 \frac{\pi}{n+1}. \end{aligned}$$

As  $n \geq 4$ , so  $n + 1 \geq 5$ . Thus

$$(2.5) \quad \begin{aligned} a_{n+1} &\geq \frac{2n}{\pi} + \frac{2}{\pi} - 0.858 \frac{\pi}{5} \\ &\geq \frac{2n}{\pi} + 0.0975. \end{aligned}$$

From (2.4) and (2.5), we get  $a_n < a_{n+1}$ .

Now we will prove that  $b_n < b_{n+1}$  for  $n \geq 3$ .

Let  $n \equiv 0 \pmod{2}$ . Then  $n + 1 \equiv 1 \pmod{2}$ . By using (2.1), we get

$$\begin{aligned} b_n &= 2 \csc \frac{\pi}{n} \leq 2 \left( \frac{1}{\frac{\pi}{n} - \frac{\pi^3}{6n^3}} \right) \\ &= \frac{2n}{\pi} \left( 1 + \frac{\pi^2}{6n^2 - \pi^2} \right) \\ &= \frac{2n}{\pi} + \frac{2n\pi}{6n^2 - \pi^2}. \end{aligned}$$

Applying Lemma 2.5, we get

$$(2.6) \quad \begin{aligned} b_n &\leq \frac{2n}{\pi} + \frac{6\pi}{6(3^2) - \pi^2} \\ &= \frac{2n}{\pi} + 0.4271. \end{aligned}$$

On the other hand, using Lemma 2.6 for  $n \geq 3$ , we have

$$\begin{aligned} b_{n+1} &= \cot \frac{\pi}{2(n+1)} \\ &\geq \left( \frac{1}{\frac{\pi}{2(n+1)}} - 0.429 \frac{\pi}{2(n+1)} \right) \\ &= \frac{2n}{\pi} + \frac{2}{\pi} - 0.429 \frac{\pi}{2(n+1)}. \end{aligned}$$

As  $n \geq 3$ , so  $n + 1 \geq 4$ . Therefore

$$\begin{aligned} (2.7) \quad b_{n+1} &\geq \frac{2n}{\pi} + \frac{2}{\pi} - 0.429 \frac{\pi}{8} \\ &\geq \frac{2n}{\pi} + 0.4682. \end{aligned}$$

From (2.6) and (2.7), we get  $b_n < b_{n+1}$ .

Now let  $n \equiv 1 \pmod{2}$ . Then  $n + 1 \equiv 0 \pmod{2}$ . By using Lemma 2.6, we get

$$\begin{aligned} (2.8) \quad b_n &= \cot \frac{\pi}{2n} \\ &\leq \frac{2n}{\pi} - \frac{\pi}{6n}. \end{aligned}$$

On the other hand, using inequality (2.1), we obtain

$$\begin{aligned} (2.9) \quad b_{n+1} &= 2 \csc \frac{\pi}{n+1} = 2 \left( \frac{1}{\sin \frac{\pi}{n+1}} \right) \\ &\geq 2 \left( \frac{1}{\frac{\pi}{n+1}} \right) = \frac{2n}{\pi} + \frac{2}{\pi} \\ &\geq \frac{2n}{\pi} + 0.6366. \end{aligned}$$

From inequalities (2.8) and (2.9), we get  $b_n < b_{n+1}$ . This proves the result.

Now we will show that the two sequences  $a_n$  and  $b_n$  are the iota energy formulas for positive and negative weight directed cycles and as iota energy is sum of absolute values of imaginary parts of eigenvalues a weighted digraph, so  $a_n$  and  $b_n$  are positive.  $\square$

Using Theorem 1.1, the characteristic polynomial of positive weight directed cycle  $C_n^+$  with weight  $r$  is given by

$$\phi_{C_n^+}(x) = x^n - r.$$

Thus, the eigenvalues of  $C_n^+$  are  $r^{\frac{1}{n}} e^{\frac{2k\pi i}{n}}$ , where  $k = 0, 1, \dots, n-1$ . Therefore, by definition of energy and iota energy, we obtain:

$$(2.10) \quad E(C_n^+) = r^{\frac{1}{n}} \sum_{k=0}^{n-1} \left| \cos \frac{2k\pi}{n} \right|,$$

$$(2.11) \quad E_c(C_n^+) = r^{\frac{1}{n}} \sum_{k=0}^{n-1} \left| \sin \frac{2k\pi}{n} \right|.$$

Bhat [4] calculated the following energy formulae for positive weight directed cycle  $C_n^+$ ,  $n \geq 2$ .

$$(2.12) \quad E(C_n^+) = \begin{cases} 2 r^{\frac{1}{n}} \cot \frac{\pi}{n} & \text{if } n \equiv 0 \pmod{4} \\ 2 r^{\frac{1}{n}} \csc \frac{\pi}{n} & \text{if } n \equiv 2 \pmod{4} \\ r^{\frac{1}{n}} \csc \frac{\pi}{2n} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Next, we calculate the iota energy formulas for positive weighted directed cycles. These formulas are similar to iota energy of positive directed cycles [19]. However, for the sake of self-containment, we include it.

**Case1:** Let  $n \equiv 0 \pmod{2}$ . Then (2.11) yields

$$\begin{aligned} E_c(C_n^+) &= r^{\frac{1}{n}} \sum_{k=0}^{n-1} \left| \sin \frac{2k\pi}{n} \right| = 2 r^{\frac{1}{n}} \left( \sum_{k=0}^{\frac{n}{2}-1} \sin \frac{2k\pi}{n} \right) \\ &= 2 r^{\frac{1}{n}} \frac{\sin \left( \frac{\pi}{4} \times \frac{2\pi}{n} \right) \sin \left( \frac{n-2}{4} \times \frac{2\pi}{n} \right)}{\sin \frac{\pi}{n}} \\ &= 2 r^{\frac{1}{n}} \frac{\sin \left( \frac{\pi}{2} - \frac{\pi}{n} \right)}{\sin \frac{\pi}{n}} = 2 r^{\frac{1}{n}} \cot \frac{\pi}{n}. \end{aligned}$$

**Case 2:** Let  $n \equiv 1 \pmod{2}$ . Then (2.11) yields

$$\begin{aligned} E_c(C_n^+) &= r^{\frac{1}{n}} \sum_{k=0}^{n-1} \left| \sin \frac{2k\pi}{n} \right| = r^{\frac{1}{n}} \left( \sum_{k=0}^{n-1} \sin \frac{k\pi}{n} \right) \\ &= r^{\frac{1}{n}} \frac{\sin \left( \frac{n}{2} \times \frac{\pi}{n} \right) \sin \left( \frac{n-1}{2} \times \frac{\pi}{n} \right)}{\sin \frac{\pi}{2n}} \\ &= r^{\frac{1}{n}} \frac{\sin \left( \frac{\pi}{2} - \frac{\pi}{2n} \right)}{\sin \frac{\pi}{2n}} = r^{\frac{1}{n}} \cot \frac{\pi}{2n}. \end{aligned}$$

Thus the iota energy of  $C_n^+$  can be written briefly as:



$$(2.13) \quad E_c(C_n^+) = \begin{cases} 2 r^{\frac{1}{n}} \cot \frac{\pi}{n} & \text{if } n \equiv 0(\text{mod}2) \\ r^{\frac{1}{n}} \cot \frac{\pi}{2n} & \text{if } n \equiv 1(\text{mod}2). \end{cases}$$

Again by using Theorem 1.1, the characteristic polynomial of negative weight directed cycle  $C_n^-$  with weight  $r$  is given by:

$$\phi_{C_n^-}(x) = x^n + |r|.$$

Thus, the eigenvalues of  $C_n^-$  are  $|r|^{\frac{1}{n}} e^{\frac{(2k+1)\pi i}{n}}$ , where  $k = 0, 1, \dots, n - 1$ . Therefore, the energy and iota energy of  $C_n^-$  are computed as:

$$(2.14) \quad E(C_n^-) = |r|^{\frac{1}{n}} \sum_{k=0}^{n-1} \left| \cos \frac{(2k+1)\pi}{n} \right|,$$

$$(2.15) \quad E_c(C_n^-) = |r|^{\frac{1}{n}} \sum_{k=0}^{n-1} \left| \sin \frac{(2k+1)\pi}{n} \right|.$$

Bhat [4] calculated the following energy formulas for negative weight directed cycle  $C_n^-$ ,  $n \geq 2$ .

$$(2.16) \quad E(C_n^-) = \begin{cases} 2 |r|^{\frac{1}{n}} \cot \frac{\pi}{n} & \text{if } n \equiv 0(\text{mod}4) \\ 2 |r|^{\frac{1}{n}} \csc \frac{\pi}{n} & \text{if } n \equiv 2(\text{mod}4) \\ |r|^{\frac{1}{n}} \csc \frac{\pi}{2n} & \text{if } n \equiv 1(\text{mod}2). \end{cases}$$

Next we calculate the formulas for iota energy of negative weight directed cycles having weight  $r$ . These formulas are similar to iota energy of negative directed cycles [11]. However, for the sake of self-containment, we include it.

**Case 1:** Let  $n \equiv 0(\text{mod } 2)$ . Then (2.15) yields

$$\begin{aligned} E_c(C_n^-) &= |r|^{\frac{1}{n}} \sum_{k=0}^{n-1} \left| \sin \frac{(2k+1)\pi}{n} \right| \\ &= 2 |r|^{\frac{1}{n}} \sum_{k=0}^{\frac{n}{2}-1} \left| \sin \frac{(2k+1)\pi}{n} \right| \\ &= 2 |r|^{\frac{1}{n}} \frac{(\sin \frac{\pi}{2})^2}{\sin \frac{\pi}{n}} = 2 |r|^{\frac{1}{n}} \csc \frac{\pi}{n}. \end{aligned}$$

**Case 2:** Let  $n \equiv 1 \pmod{2}$ . Then (2.15) yields

$$\begin{aligned}
 E_c(C_n^-) &= |r|^{\frac{1}{n}} \sum_{k=0}^{n-1} \left| \sin \frac{(2k+1)\pi}{n} \right| \\
 &= |r|^{\frac{1}{n}} \sum_{k=0}^{\frac{n}{2}-1} \left| \sin \frac{k\pi}{n} \right| \\
 &= |r|^{\frac{1}{n}} \frac{\sin \frac{n\pi}{2n} \sin (n-1) \frac{\pi}{2n}}{\sin \frac{\pi}{2n}} \\
 &= |r|^{\frac{1}{n}} \frac{\sin \left( \frac{\pi}{2} - \frac{\pi}{2n} \right)}{\sin \frac{\pi}{2n}} = |r|^{\frac{1}{n}} \cot \frac{\pi}{2n}.
 \end{aligned}$$

In brief, we write:

$$(2.17) \quad E_c(C_n^-) = \begin{cases} 2 |r|^{\frac{1}{n}} \csc \frac{\pi}{n} & \text{if } n \equiv 0 \pmod{2} \\ |r|^{\frac{1}{n}} \cot \frac{\pi}{2n} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

A unicyclic weighted digraph is a weighted digraph in which number of vertices equals number of weighted arcs and there is only one cycle. By cycle-weight of a unicyclic weighted digraph we mean product of weights of arcs of a cycle. Khan et al. [19] finds the unicyclic digraphs with minimal and maximal iota energy among all the unicyclic digraphs. Bhat [4] finds the minimal and maximal energy of unicyclic weighted digraphs with cycle weight  $r \in [-1, 1] \setminus \{0\}$ . Now we characterize the unicyclic weighted digraphs with minimal and maximal iota energy in the set of unicyclic weighted digraphs of order  $n$  with cycle weight  $r \in [-1, 1] \setminus \{0\}$ .

For a fixed real number  $r \neq 0$ , let  $\mathbb{U}_n(r)$  denote the set of unicyclic weighted digraphs of order  $n$  and cycle weight  $r$ . Also for a fixed real number  $r \neq 0$ , let  $\mathbb{C}_n(r)$  denote the set whose elements are cycles of order  $n$  and weight  $r$ . Note that  $\mathbb{C}_n(r) \subset \mathbb{U}_n(r)$ .

**Theorem 2.8.** *Among all unicyclic weighted digraphs in  $\mathbb{U}_n(r)$ , where  $-1 \leq r < 0$ , every element of  $\mathbb{C}_n(r)$  has maximal iota energy. Moreover, minimal iota energy is attained in all those  $D \in \mathbb{U}_n(r)$  which contain a cycle of length 3.*

*Proof.* Let  $D \in \mathbb{U}_n(r)$  with a cycle of length  $m \leq n$ . By Theorem 2.4,  $E_c(D) = E_c(C_m^-)$ . It can easily be seen that the sequence  $\{|r|^{\frac{1}{m}}\}$  is strictly increasing for  $0 < |r| < 1$  and constant for  $r = -1$ . As the sequence  $b_m$  in (2.17) is positive, so by equation (2.17) and Lemma 2.7, the sequence  $\langle |r|^{\frac{1}{m}} b_m \rangle$  is strictly increasing. From this, the result follows. □

The next theorem gives the weighted digraphs in  $\mathbb{U}_n(r)$ , where  $0 < r \leq 1$  with minimal and maximal iota energy.

**Theorem 2.9.** *Among all unicyclic weighted digraphs in  $\mathbb{U}_n(r)$ , where  $0 < r \leq 1$ , every element of  $\mathbb{C}_n(r)$  has maximal iota energy. Moreover, minimal iota energy is attained in all those  $D \in \mathbb{U}_n(r)$  which contain a cycle of length 2.*

*Proof.* Let  $D \in \mathbb{U}_n(r)$ , with a cycle of length  $m \leq n$ . By Theorem 2.4,  $E_c(D) = E_c(C_m^+)$ . It can be easily seen that the sequence  $\langle r^{\frac{1}{m}} \rangle$  is strictly increasing for  $0 < r \leq 1$  and constant for  $r = 1$ . As the sequence  $a_m$  in (2.13) is positive, so by equation (2.13) and Lemma 2.7, the sequence  $\langle r^{\frac{1}{m}} a_m \rangle$  is strictly increasing. This proves the result.  $\square$

Now, we are in a position to determine weighted digraphs with minimal and maximal iota energy in the set of unicyclic weighted digraphs by varying cycle-weight  $r$ . From energy formulae (2.13) and (2.17) of weighted directed cycles, we have the following result.

**Theorem 2.10.** *Among all unicyclic weighted digraphs with  $n$  vertices and cycle weight  $r$  with  $0 < |r| \leq 1$ , maximal iota energy is attained by*

- (i) every element of  $\mathbb{C}_n(-1)$  if  $n \equiv 0 \pmod{2}$ .
- (ii) every element of  $\mathbb{C}_n(-1)$  and  $\mathbb{C}_n(1)$  if  $n \equiv 1 \pmod{2}$ .

*Moreover, minimal iota energy is attained by all those unicyclic weighted digraphs which contain a positive weight directed cycle  $C_2^+$  with weight  $r$ , where  $0 < r \leq 1$ .*

Khan et al. [19] compares the energy and iota energy of directed cycles. In the following theorem, we compare the energy and iota energy of positive weight directed cycles with weight  $r > 0$ . The results are similar to results of Khan et al. [19]

**Theorem 2.11.** *Let  $n \geq 2$  be an integer. Then energy and iota energy of positive weight directed cycle  $C_n^+$  satisfies the following:*

- (i)  $E(C_n^+) = E_c(C_n^+)$  if and only if  $n \equiv 0 \pmod{4}$ .
- (ii)  $E(C_n^+) > E_c(C_n^+)$  if and only if  $n \equiv 2 \pmod{4}$  or  $n \equiv 1 \pmod{2}$ .

*Proof.* Using Equations (2.12) and (2.13), we easily get the desired results.  $\square$

For any weighted unicyclic digraph it is easy to see that iota energy of weighted digraph is equal to iota energy of its weighted directed cycle. The following corollary is an immediate consequence of Theorem 2.11.

**Corollary 2.12.** *Let  $D$  be an  $n$ -vertex unicyclic weighted digraph with unique positive weight directed cycle  $C_m^+$ ,  $2 \leq m \leq n$ . Then the following hold:*

- (i)  $E(D) = E_c(D)$  if and only if  $m \equiv 0 \pmod{4}$ .
- (ii)  $E(D) > E_c(D)$  if and only if  $m \equiv 2 \pmod{4}$  or  $m \equiv 1 \pmod{2}$ .

In the next theorem, we give comparison between the energy and iota energy of negative weight directed cycles with weight  $r < 0$ .

**Theorem 2.13.** *Let  $n \geq 2$  be an integer. Then energy and iota energy of negative weight directed cycle  $C_n^-$  satisfies the following:*

- (i)  $E(C_n^-) = E_c(C_n^-)$  if and only if  $n \equiv 0 \pmod{4}$ .
- (ii)  $E(C_n^-) < E_c(C_n^-)$  if and only if  $n \equiv 2 \pmod{4}$ .
- (iii)  $E(C_n^-) > E_c(C_n^-)$  if and only if  $n \equiv 1 \pmod{2}$ .

*Proof.* Using Equations (2.16) and (2.17), we easily get the desired results.  $\square$

The following corollary is an immediate consequence of Theorem 2.13.

**Corollary 2.14.** *Let  $D$  be an  $n$ -vertex unicyclic weighted digraph with unique negative weight directed cycle  $C_m^-$ ,  $2 \leq m \leq n$ . Then the following hold:*

- (i)  $E(D) = E_c(D)$  if and only if  $m \equiv 0 \pmod{4}$ .
- (ii)  $E(D) < E_c(D)$  if and only if  $m \equiv 2 \pmod{4}$ .
- (iii)  $E(D) > E_c(D)$  if and only if  $m \equiv 1 \pmod{2}$ .

Using formulas (2.12), (2.16), (2.13) and (2.17), Theorem 2.15 gives us relation between iota energy of positive weight and negative weight directed cycles.

**Theorem 2.15.** *Let  $w(C_n^+) = r_1$  and  $w(C_n^-) = r_2$ , where  $r_1, r_2 \in \mathbb{R} \setminus \{0\}$ . The iota energy of positive and negative weight directed cycles satisfies the following:*

*For  $n \equiv 1 \pmod{2}$ , we obtain:*

- (i)  $E_c(C_n^+) = E_c(C_n^-)$  if and only if  $r_1 = |r_2|$ .
- (ii)  $E_c(C_n^+) > E_c(C_n^-)$  if and only if  $r_1 > |r_2|$ .
- (iii)  $E_c(C_n^+) < E_c(C_n^-)$  if and only if  $r_1 < |r_2|$ .

*For  $n \equiv 0 \pmod{2}$ , we obtain:*

- (iv)  $E_c(C_n^-) > E_c(C_n^+)$  if and only if  $r_1 \leq |r_2|$ .

### 3. Integral representation and upper bound for the iota energy of weighted digraphs

Integral representation of energy is very useful as one can find energy of digraph without finding the zeros of its characteristic polynomial. The energy of digraphs in integral form was given by Pe na and Rada [22]. Bhat [4] represented the energy of weighted digraphs in integral form. The following theorem is given by Bhat [4].

**Theorem 3.1.** (Bhat [4]) *Let  $D$  is a weighted digraph of degree  $n$ . Let  $z_1, \dots, z_n$  be its eigenvalues. Then*

$$E(D) = \sum_{k=1}^n |\operatorname{Re}(z_k)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n - \frac{i\lambda\Phi'_D(i\lambda)}{\Phi_D(i\lambda)} \right) d\lambda,$$

where  $\operatorname{Re}(z_k)$  denotes the real part of eigenvalue  $z_k$  and  $i = \sqrt{-1}$ .

In the following theorem, we represent iota energy of weighted digraphs in integral form. Proof is analogous to the proof of Theorem 3.3 [11].

**Theorem 3.2.** *Let  $D$  be an  $n$ -vertex weighted digraph with characteristic polynomial  $\Phi_D(\lambda)$ . Let  $z_1, \dots, z_n$  be its eigenvalues. Then*

$$E_c(D) = \sum_{k=1}^n |\text{Im}(z_k)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n - \frac{\lambda \Phi'_D(\lambda)}{\Phi_D(\lambda)} \right) d\lambda,$$

where  $\text{Im}(z_k)$  is the imaginary part of  $z_k$ .

**Example 3.3.** *Consider the weighted digraph  $D$  as shown in Figure 2. The characteristic polynomial of  $D$  is given by  $\Phi_D(\lambda) = \lambda^4 + 6\lambda^2 + 9$  and hence*

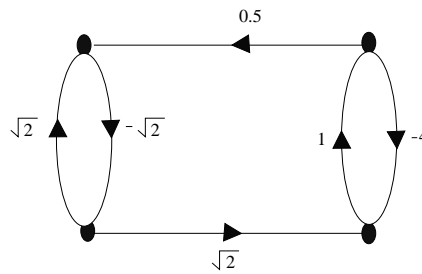


FIGURE 2.

$$\begin{aligned} E_c(D) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ 4 - \frac{\lambda(4\lambda^3 + 12\lambda)}{\lambda^4 + 6\lambda^2 + 9} \right] d\lambda = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{12\lambda^2 + 36}{\lambda^4 + 6\lambda^2 + 9} \right) d\lambda \\ &= \frac{2}{\pi} \int_{-\infty}^{\infty} \left( \frac{6}{\lambda^2 + 3} \right) d\lambda = \frac{12}{\pi} \left[ \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{\lambda}{\sqrt{3}} \right) \right]_{-\infty}^{\infty} \\ &= \frac{12}{\pi} \left( \frac{\pi}{\sqrt{3}} \right) = \frac{12}{\sqrt{3}} = 4\sqrt{3}. \end{aligned}$$

An alternating sequence of vertices and directed arcs is called a directed walk. Let  $w^+(2)$  and  $w^-(2)$  respectively denote the number of positive and negative closed directed walks of length 2. A square matrix  $M$  is called normal matrix if  $M^T M = M M^T$ , where  $M^T$  denotes the transpose of a matrix  $M$ . The following Lemma given by Bhat [4] is used to find this bound.

**Lemma 3.4.** (Bhat [4]) *Let  $D$  be a weighted digraph having  $n$  vertices and  $m$  arcs and let  $\lambda_1, \dots, \lambda_n$  be its eigenvalues. Then*

- (i)  $\sum_{k=1}^n (\text{Re}(\lambda_k))^2 - \sum_{k=1}^n (\text{Im}(\lambda_k))^2 = w^+(2) - w^-(2)$ .
- (ii)  $\sum_{k=1}^n (\text{Re}(\lambda_k))^2 + \sum_{k=1}^n (\text{Im}(\lambda_k))^2 \leq \sum_{k=1}^m (w(e_k))^2$  with equality if and only if  $D$  is normal, that is,  $A(D)$  is a normal matrix.

A weighted digraph  $D$  is said to be symmetric if  $uv \in A$  with weight  $r$ , then  $vu \in A$  with same weight  $r$ . We denote by  $r \overleftrightarrow{K}_2$ ,  $r \in \mathbb{R} \setminus \{0\}$ , a pair of symmetric arcs  $uv$  and  $vu$  both with weight  $r$  and by  $q \oplus (r \overleftrightarrow{K}_2)$ , direct sum of  $q$  copies of  $(r \overleftrightarrow{K}_2)$ . Now we will give iota energy bound. The proof of next theorem is similar to the proof of Theorem 5.4 in [4]. However for the sake of self containment, we include the proof.

**Theorem 3.5.** *Let  $D$  be a weighted digraph with  $n$  vertices and  $m$  arcs. Then*

$$E_c(D) \leq \sqrt{\frac{n}{2} \left( \sum_{k=1}^m (w(e_k))^2 - w^+(2) + w^-(2) \right)}.$$

*Proof.* From (i) of Lemma 3.4, we have

$$(3.1) \quad \sum_{k=1}^n (\text{Im}(\lambda_k))^2 = \sum_{k=1}^n (\text{Re}(\lambda_k))^2 - w^+(2) + w^-(2).$$

By subtracting (i) from (ii) of Lemma 3.4, we get

$$(3.2) \quad \sum_{k=1}^n (\text{Im}(\lambda_k))^2 \leq \frac{1}{2} \left( \sum_{k=1}^m (w(e_k))^2 - w^+(2) + w^-(2) \right).$$

Applying the Cauchy-Schwarz inequality to vectors  $(|\text{Im}(\lambda_1)|, \dots, |\text{Im}(\lambda_n)|)$  and  $(1, 1, \dots, 1) \in \mathbb{R}^n$ , we have

$$(3.3) \quad \begin{aligned} E_c(D) &= \sum_{k=1}^n |\text{Im}(\lambda_k)| \leq \sqrt{n} \sqrt{\sum_{k=1}^n (\text{Im} \lambda_k)^2} \\ &\leq \sqrt{n} \sqrt{\frac{1}{2} \left( \sum_{k=1}^m (w(e_k))^2 - w^+(2) + w^-(2) \right)} \\ &= \sqrt{\frac{n}{2} \left( \sum_{k=1}^m (w(e_k))^2 - w^+(2) + w^-(2) \right)}. \end{aligned}$$

It gives the required result. □

**Remark 3.6.** *The upper bound in Theorem 3.5 is attained by weighted digraphs  $D_1 = \frac{n}{2} \oplus (r \overleftrightarrow{K}_2, +)$  and  $D = \frac{n}{2} \oplus (r \overleftrightarrow{K}_2, -)$ , where  $(r \overleftrightarrow{K}_2, +)$  and  $(r \overleftrightarrow{K}_2, -)$  respectively denote symmetric weighted digraphs obtained from  $K_2^+$  and  $K_2^-$  having weight  $r$ .*

#### 4. Equienergetic weighted digraphs

Two weighted digraphs are isomorphic if their underlying graphs are isomorphic such that the weights are preserved. Two weighted digraphs are said to be cospectral if they have the same spectrum, otherwise noncospectral. Two weighted digraphs of same order having the same iota energy are said to be equienergetic. Obviously cospectral and isomorphic weighted digraphs of the same order are always equienergetic. Therefore the problem reduces to find noncospectral pairs of equienergetic weighted

digraphs. So, in this section, we are interested in constructing a class of equienergetic noncospectral weighted digraphs. Let  $D$  be a weighted digraph and let  $D_1, \dots, D_r$  be its strong components. Then, from the proof of Theorem 2.4, the characteristic polynomial of  $D$  is given by:

$$(4.1) \quad \Phi_D(\lambda) = \Phi_{D_1}(\lambda) \cdots \Phi_{D_r}(\lambda).$$

The following lemma is about the characteristic polynomial of weighted digraph  $D$  such that all of its weighted cycles are vertex-disjoint.

**Lemma 4.1.** *Let  $D$  be an  $n$ -vertex weighted digraph containing cycles  $C_1, \dots, C_r$  of lengths  $m_1, m_2, \dots, m_r$ , respectively. Assume that  $C_1, \dots, C_r$  are pairwise vertex-disjoint. Then*

$$\Phi_D(\lambda) = \lambda^{n-m} \Phi_{C_1}(\lambda) \cdots \Phi_{C_r}(\lambda).$$

where  $m = \sum_{i=1}^r m_i$ .

*Proof.* The proof follows from (4.1). □

**Lemma 4.2.** *Let  $n \geq 4$  be an even positive integer. Let  $w(C_n^+) = r_1$ ,  $w(C_{\frac{n}{2}}^-) = r_2$  and  $r_1 = |r_2|^2$ . Then  $E_c(C_n^+) = 2 E_c(C_{\frac{n}{2}}^-)$  if and only if  $n \equiv 2 \pmod{4}$ .*

*Proof.* ( $\implies$ ) Suppose that  $E_c(C_n^+) = 2 E_c(C_{\frac{n}{2}}^-)$ . On contrary suppose that  $n \not\equiv 2 \pmod{4}$ . Then,  $n \equiv 0 \pmod{4}$  and  $\frac{n}{2} \equiv 0 \pmod{2}$ . By (2.13) and (2.17), we get

$$E_c(C_n^+) = 2 r_1^{\frac{1}{n}} \cot \frac{\pi}{n},$$

$$E_c(C_{\frac{n}{2}}^-) = 2 |r_2|^{\frac{2}{n}} \csc \frac{\pi}{n}.$$

The above two equations together with the equation  $E_c(C_n^+) = 2 E_c(C_{\frac{n}{2}}^-)$  and  $r_1 = |r_2|^2$  yield

$$\cot \frac{\pi}{n} = 2 \csc \frac{\pi}{n}$$

Using the trigonometric identities, we get  $\cos \frac{\pi}{n} = 2$ . This, however is not possible. Hence  $n \equiv 2 \pmod{4}$ .

( $\impliedby$ ) Suppose that  $n \equiv 2 \pmod{4}$ . Then  $\frac{n}{2} \equiv 1 \pmod{2}$ . By (2.13) and (2.17), we have

$$E_c(C_n^+) = 2 r_1^{\frac{1}{n}} \cot \frac{\pi}{n},$$

$$E_c(C_{\frac{n}{2}}^-) = |r_2|^{\frac{2}{n}} \cot \frac{\pi}{n}.$$

The above two equations with  $r_1 = |r_2|^2$  yields

$$E_c(C_n^+) = 2 E_c(C_{\frac{n}{2}}^-).$$

This completes the proof. □

**Lemma 4.3.** *Let  $n \geq 3$  be an odd positive integer. Let  $w(C_n^+) = r_1$ ,  $w(C_n^-) = r_3$  and  $r_1 = |r_3|$ . Then  $E_c(C_n^+) = E_c(C_n^-)$  if and only if  $n \equiv 1 \pmod{2}$ .*

*Proof.* ( $\implies$ ) Suppose that  $E_c(C_n^+) = E_c(C_n^-)$ . On contrary, suppose that  $n \not\equiv 1 \pmod{2}$ . Then,  $n \equiv 0 \pmod{2}$ . By (2.13) and (2.17), we get

$$E_c(C_n^+) = 2 r_1^{\frac{1}{n}} \cot \frac{\pi}{n},$$

$$E_c(C_n^-) = 2 |r_3|^{\frac{1}{n}} \csc \frac{\pi}{n}.$$

The above two equations together with  $E_c(C_n^+) = E_c(C_n^-)$  and  $r_1 = |r_3|$  yield

$$\cot \frac{\pi}{n} = \csc \frac{\pi}{n}$$

Using the trigonometric identities, we get  $\cos \frac{\pi}{n} = 1$ , that is,  $\frac{\pi}{n} = 2k\pi$ , where  $k$  is an integer. This, however is not possible. Hence  $n \equiv 1 \pmod{2}$ .

( $\impliedby$ ) Suppose that  $n \equiv 1 \pmod{2}$ . By (2.13) and (2.17), we have

$$E_c(C_n^+) = r_1^{\frac{1}{n}} \cot \frac{\pi}{2n},$$

$$E_c(C_n^-) = |r_3|^{\frac{1}{n}} \cot \frac{\pi}{2n}.$$

The above two equations with  $r_1 = |r_3|$  gives

$$E_c(C_n^+) = E_c(C_n^-).$$

This completes the proof. □

The next two theorems gives a class of noncospectral equienergetic weighted digraphs.

**Theorem 4.4.** *Let  $D$  be an  $n$ -vertex weighted digraph,  $n \geq 4$ , with  $k$  vertex-disjoint positive weight directed cycles of lengths  $m_1, \dots, m_k$  and weights  $w_1, \dots, w_k$ , where  $m_j \equiv 2 \pmod{4}$ ,  $j = 1, 2, \dots, k$ . Take an  $n$ -vertex weighted digraph  $H$  with  $2k$  vertex-disjoint negative weight directed cycles of lengths  $\frac{m_1}{2}, \frac{m_1}{2}, \dots, \frac{m_k}{2}, \frac{m_k}{2}$  with weights  $r_1, r_1, \dots, r_k, r_k$ , where  $w_j = |r_j|^2$ ,  $j = 1, 2, \dots, k$ . Then  $D$  and  $H$  are noncospectral equienergetic weighted digraphs.*

*Proof.* By Theorem 1.1 and Lemma 4.1, the characteristic polomila of  $D$  and  $H$  are respectively,

$$(4.2) \quad \Phi_D(x) = \prod_{j=1}^k x^{n-q} (x^{m_j} - w_j)^k,$$

$$(4.3) \quad \Phi_H(x) = \prod_{j=1}^k x^{n-q} \left( x^{\frac{m_j}{2}} + |r_j| \right)^{2k},$$

where  $q = \sum_{j=1}^k m_j$ . It is evident that the roots of both polynomials in (4.2) and (4.3) are not the same.

Thus  $D$  and  $H$  are noncospectral. By Lemma 4.2 and Theorem 2.4, we have

$$E_c(D) = \sum_{j=1}^k E_c(C_{m_j}^+) = 2 \sum_{j=1}^k E_c(C_{\frac{m_j}{2}}^-)$$

$$= E_c(H).$$



This completes the proof. □

**Theorem 4.5.** *Let  $D$  be an  $n$ -vertex weighted digraph,  $n \geq 3$ , with  $k$  vertex-disjoint directed positive weight cycles of lengths  $m_1, \dots, m_k$  and weights  $w_1, \dots, w_k$ , where  $m_j \equiv 1 \pmod{2}$ ,  $j = 1, 2, \dots, k$ . Take an  $n$ -vertex weighted digraph  $H$  with  $k$  vertex-disjoint directed negative weight cycles of lengths  $m_1, \dots, m_k$  with weights  $r_1, \dots, r_k$ , where  $r_j \equiv 1 \pmod{2}$  and  $w_j = |r_j|$ ,  $j = 1, 2, \dots, k$ . Then  $D$  and  $H$  are noncospectral equienergetic weighted digraphs.*

*Proof.* Proof follows from Lemma 4.3 □

Theorem 4.6 gives a weighted digraph whose iota energy is product of twice the number of its vertices and weight of  $C_2^-$ .

**Theorem 4.6.** *Let  $D$  be a weighted digraph of order  $n$  having eigenvalues  $\lambda_1, \dots, \lambda_n$  and let  $w(D) = w$  such that if  $D$  is positive then  $0 < w \leq 1$  and if  $D$  is negative then  $0 < |w| \leq 1$  and  $|\text{Im}(\lambda_k)| \leq 1$  for every  $k = 1, 2, \dots, n$ . Let  $w(C_2^-) = r$  and  $|r| \geq 1$ . Then,*

$$E_c(D \times C_2^-) = 2n|r|^{\frac{1}{2}}.$$

*Proof.* Let  $\lambda_1, \dots, \lambda_q$  be eigenvalues with non-negative imaginary part and  $\lambda_{q+1}, \dots, \lambda_n$  be those with negative imaginary part. Eigenvalues of cartesian product of  $D \times C_2^-$  are  $\lambda_1 \pm i|r|^{\frac{1}{2}}, \dots, \lambda_q \pm i|r|^{\frac{1}{2}}, \lambda_{q+1} \pm i|r|^{\frac{1}{2}}, \dots, \lambda_n \pm i|r|^{\frac{1}{2}}$ . Therefore,

$$(4.4) \quad E_c(D \times C_2^-) = \sum_{k=1}^q (|\text{Im}(\lambda_k) + |r|^{\frac{1}{2}}| + |\text{Im}(\lambda_k) - |r|^{\frac{1}{2}}|) + \sum_{k=q+1}^n (|\text{Im}(\lambda_k) + |r|^{\frac{1}{2}}| + |\text{Im}(\lambda_k) - |r|^{\frac{1}{2}}|)$$

As  $\text{Im}(\lambda_k) \leq 1$  for all  $k = 1, 2, \dots, n$  and  $|r|^{\frac{1}{2}} \geq 1$ , it follows that

$$\begin{aligned} E_c(D \times C_2^-) &= \sum_{k=1}^q (\text{Im}(\lambda_k) + |r|^{\frac{1}{2}} + |r|^{\frac{1}{2}} - \text{Im}(\lambda_k)) + \sum_{k=q+1}^n (|r|^{\frac{1}{2}} - \text{Im}(\lambda_k) + \text{Im}(\lambda_k) + |r|^{\frac{1}{2}}) \\ &= \sum_{k=1}^q 2|r|^{\frac{1}{2}} + \sum_{k=q+1}^n 2|r|^{\frac{1}{2}} \\ &= |r|^{\frac{1}{2}} \left( \sum_{k=1}^q 2 + \sum_{k=q+1}^n 2 \right) \\ &= |r|^{\frac{1}{2}} (2q + 2(n - q)) = 2n|r|^{\frac{1}{2}}. \end{aligned}$$

This completes the proof. □

Next corollary gives another class of noncospectral equienergetic weighted digraphs.

**Corollary 4.7.** *For  $n \geq 2$ , Let  $w(C_n^+) = r_1$ ,  $w(C_n^-) = r_4$  and  $0 < r_1, |r_4| \leq 1$ . Then*

$$E_c(C_n^- \times C_2^-) = E_c(C_n^+ \times C_2^-).$$

Moreover,  $C_n^- \times C_2^-$  and  $C_n^+ \times C_2^-$  are non-cospectral weighted digraphs.

*Proof.* We know that eigenvalues of  $C_n^-$  are  $|r_4|^{\frac{1}{n}} \exp \frac{i(2k+1)\pi}{n}$ ,  $k = 0, 1, \dots, n-1$  and those of  $C_n^+$  are  $r_1^{\frac{1}{n}} \exp \frac{2ik\pi}{n}$ ,  $k = 0, 1, \dots, n-1$ . One can easily see that eigenvalues of  $C_n^-$  and  $C_n^+$  meet requirement of Theorem 4.6. So

$$E_c(C_n^- \times C_2^-) = E_c(C_n^+ \times C_2^-) = 2n|r|^{\frac{1}{2}}.$$

Moreover,  $r_1^{\frac{1}{n}} + i|r|^{\frac{1}{2}} \in \text{Spec}(C_n^+ \times C_2^-)$  but  $r_1^{\frac{1}{n}} + i|r|^{\frac{1}{2}} \notin \text{Spec}(C_n^- \times C_2^-)$  implying that  $C_n^- \times C_2^-$  and  $C_n^+ \times C_2^-$  are noncospectral. Both the weighted digraphs have  $2n$  vertices follows by definition of cartesian product.  $\square$

**Theorem 4.8.** *Let  $D$  be a weighted digraph of order  $n$  having eigenvalues  $\lambda_1, \dots, \lambda_n$  and let  $w(D) = w$  such that if  $D$  is positive then  $0 < w \leq 1$  and if  $D$  is negative then  $0 < |w| \leq 1$  and  $|\text{Im}(\lambda_k)| \leq 1$  for every  $k = 1, 2, \dots, n$ . Let  $w(C_2^+) = r_4$*

$$E_c(D \times C_2^+) = 2E_c(D).$$

*Proof.* Proof is similar to the proof of Theorem 5.4.  $\square$

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