



ON THE MINIMUM STOPPING SETS OF PRODUCT CODES

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ABSTRACT. It is shown that the certain combinatorial structures called stopping sets have the important role in analysis of iterative decoding. In this paper, the number of minimum stopping sets of a product code is determined by the number of the minimum stopping sets of the corresponding component codes. As an example, the number of minimum stopping sets of the r -dimensional SPC product code is computed.

1. Introduction

Let F_2 be the field of 2 elements. A binary linear $[n, k]$ code C is a k -dimensional vector subspace of the vector space F_2^n , where $F_2^n = \{ (x_0, x_1, \dots, x_{n-1}) \mid x_i \in F_2 \}$. The elements of C are called codewords and the Hamming weight of a codeword is the number of non-zero coordinates. An $[n, k, d]$ code is an $[n, k]$ code with minimum (non-zero) Hamming weight d . A generator matrix for a code C is any $k \times n$ matrix G whose rows form a basis for C . The dual code C^\perp of C is defined as $C^\perp = \{ x \in F_2^n \mid x \cdot c = 0 \text{ for all } c \in C \}$. An $(n - k) \times n$ generator matrix H of C^\perp is called a parity check matrix of C .

Recently, it has shown that the performance of linear codes under iterative decoding over the BEC is determined by certain combinatorial structures called stopping sets [8]. Let us define the concept of a stopping set in the next section. The size of the smallest non-empty stopping set in a parity-check matrix H of C is called the stopping distance of H . This later concept is denoted by $s(H)$. It was shown that the problem of determining the stopping distance of a parity-check matrix is a NP-hard problem [3]. The number of stopping sets of H of size i is denoted by $S_i(H)$.

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One method for constructing powerful codes from other codes is the product coding technique. Product codes were introduced by Elias [2] in 1954. These codes are a subclass of concatenated codes. Consider two binary linear codes R and C with parameters $[n, k, d]$ and $[n', k', d']$, respectively. The product code $P = R \otimes C$ consists of all the matrices with all rows in the code R and all columns in the code C . The component codes R and C are called the row code and the column code, respectively. The product code P is a binary linear $[nn', kk', dd']$ code. The minimum codewords of the product code $R \otimes C$ are determined by the minimum codewords of the component codes R and C [5]. The number of the minimum stopping sets has a crucial role in performance of C under iterative decoding akin to the role of number of the minimum codewords of C under maximum-likelihood decoding over BEC when the erasure probability is small [10, 11]. In this paper the number of minimum stopping sets of the product code $R \otimes C$ is determined by the number of the minimum stopping sets of the corresponding component codes.

The paper is organized as follows. The number of minimum stopping sets of a product code is determined in Section 2. Computing of the number of minimum stopping sets of the r -dimensional SPC product code, is given in Section 3.

2. Minimum Stopping Sets of product codes

Let C be a binary linear $[n, k, d]$ code and let $H = (h_{ij})$ be a parity-check matrix of C . We index the columns of H by $1, 2, \dots, n$ and consider S as a subset of $\{1, 2, \dots, n\}$. Let $N_r(H)$ denote the number of rows of H and H_S denote the $N_r(H) \times |S|$ submatrix of H consisting of columns indexed by S .

A set S is the support of a codeword if and only if all rows in H_S have even weight, i.e.

$$|\{j \in S : h_{ij} = 1\}| = 0 \pmod{2} \quad \forall i = 1, 2, \dots, N_r(H).$$

Definition 2.1. For a given parity-check matrix H of C , a subset S of column indices of H is called a *stopping set* in H if H_S has no row of weight one, i.e.

$$|\{j \in S : h_{ij} = 1\}| \neq 1 \quad \forall i = 1, 2, \dots, N_r(H).$$

The size of a stopping set S is equal to the number of elements in S . A stopping set of the smallest size is called a *minimum stopping set*. It follows from definition that the support of any codeword is a stopping set. Hence $s(H) \leq d$.

In this paper H_R and H_C stand for parity-check matrices of two binary linear codes R and C with parameters $[n, k, d]$ and $[n', k', d']$, respectively. It is known that the product code $R \otimes C$ has the following parity-check matrix H_P [9] wherein “ \otimes ” stands for the Kronecker product operation. Note that the rows of H_P may be linearly dependent.

$$(2.1) \quad H_P = \begin{pmatrix} H_R \otimes I_{n'} \\ I_n \otimes H_C \end{pmatrix}$$

Definition 2.2. Consider the product code $R \otimes C$ defined by two given codes R and C of lengths n and n' , respectively. The set $B_i := \{i, i + n', \dots, i + (n - 1)n'\}$, $1 \leq i \leq n'$, is called the i th

row-set and the set $B'_j := \{(j - 1)n' + 1, (j - 1)n' + 2, \dots, (j - 1)n' + n'\}$, $1 \leq j \leq n$, is called the j th column-set. Let E be a subset of $\{1, 2, \dots, nn'\}$. For each row-set B_i and column-set B'_j we set $E_i := \{(a - i)/n' + 1 : a \in B_i \cap E\}$ and $E'_j := \{a - (j - 1)n' : a \in B'_j \cap E\}$; note that $(a - i)/n'$ is a nonnegative integer.

The stopping sets in the parity-check matrix H_P of the product code $R \otimes C$ are determined by stopping sets in the parity-check matrices H_R and H_C of the corresponding component code R and C , respectively [6].

Lemma 2.3. [6] A subset E of $\{1, 2, \dots, nn'\}$ is a stopping set in H_P , given by (2.1), if and only if for each row-set B_i with $B_i \cap E \neq \emptyset$, E_i is a stopping set in H_R and for each column-set B'_j satisfying $B'_j \cap E \neq \emptyset$, E'_j is a stopping set in H_C .

Lemma 2.4. [6] Let $S_R = \{i_1, i_2, \dots, i_r\}$ and $S_C = \{j_1, j_2, \dots, j_c\}$ be two stopping sets in the parity-check matrices H_R and H_C , respectively. Then the set S_{RC} defined by (2.2) is a stopping set in H_P .

$$(2.2) \quad S_{RC} = \{(i_1 - 1)n' + j_1, (i_1 - 1)n' + j_2, \dots, (i_1 - 1)n' + j_c, \\ (i_2 - 1)n' + j_1, (i_2 - 1)n' + j_2, \dots, (i_2 - 1)n' + j_c, \\ \dots, (i_r - 1)n' + j_1, (i_r - 1)n' + j_2, \dots, (i_r - 1)n' + j_c\}.$$

The stopping set in form (2.2) is called the *obvious stopping set* of the product code. The size of S_{RC} is equal to the product of the size of stopping sets S_R and S_C .

Example 2.5. Let R be the $[2, 1, 2]$ binary code and C be the $[8, 4, 4]$ Reed-Muller code with parity-check matrix H_C :

$$(2.3) \quad H_C = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$H_P = \begin{pmatrix} I_8 & I_8 \\ H_C & \mathbf{0} \\ \mathbf{0} & H_C \end{pmatrix}.$$

Setting $E = \{1, 3, 4, 9, 11, 12\}$, we have

$$E_1 = E_3 = E_4 = \{1, 2\}, \\ E_2 = E_5 = E_6 = E_7 = E_8 = \emptyset, \\ E'_1 = E'_2 = \{1, 3, 4\}.$$

The set $\{1, 2\}$ is a stopping set in H_R and the set $\{1, 3, 4\}$ is a stopping set in H_C . Hence Lemma 2.3 implies that $\{1, 3, 4, 9, 11, 12\}$ is a stopping set in H_P . This stopping set is a obvious stopping set.

Lemma 2.6. Let the stopping distance of parity-check matrices H_R and H_C be $s(H_R)$ and $s(H_C)$, respectively. Let E be a stopping set in H_P .

- (i) If the size of $\cup_{i=1}^{n'} E_i$ is equal to $s(H_R)$. Then E is the obvious stopping set.

(ii) If the size of $\cup_{j=1}^n E'_j$ is equal to $s(H_C)$. Then E is the obvious stopping set.

Proof. Consider case (i). Suppose that the size of $\cup_{i=1}^{n'} E_i$ is equal to $s(H_R)$. Since E is a stopping set in H_P , Lemma 2.3 implies that for each row-set B_i with $B_i \cap E \neq \emptyset$, E_i is a stopping set in H_R . Hence the size of E_i where $B_i \cap E \neq \emptyset$, is at least $s(H_R)$. Thus if $E_i \neq \emptyset$, $1 \leq i \leq n'$, the size of E_i is equal to $s(H_R)$. Therefore for each row-set B_i with $B_i \cap E \neq \emptyset$, we have $E_i = \cup_{i=1}^{n'} E_i$. Hence for i , $1 \leq i \leq n'$, the set E_i is empty or the constant set $\cup_{i=1}^{n'} E_i$. Therefore E is an obvious stopping set. Statement (ii) is proved in a similar way. \square

It is shown that the stopping distance of the parity-check matrix H_P of the product code $R \otimes C$ is equal to the product of the stopping distances of parity-check matrices H_R and H_C of component codes R and C [6].

Lemma 2.7. [6] Let the stopping distance of parity-check matrices H_R and H_C be $s(H_R)$ and $s(H_C)$, respectively. Then the stopping distance of H_P is $s(H_R)s(H_C)$.

It is well known that the performance under iterative decoding over the BEC is computed by the number of minimum stopping sets when the erasure probability is small [10, 11].

Theorem 2.8. Let the number stopping sets of size $s(H_R)$ and $s(H_C)$ in parity-check matrices H_R and H_C be $S_{s(H_R)}$ and $S_{s(H_C)}$, respectively. Then the number of stopping sets of size $s(H_P)$ in H_P is $S_{s(H_R)}S_{s(H_C)}$.

Proof. Suppose $E \subseteq \{1, 2, \dots, nn'\}$ is a stopping set of size $s(H_R)s(H_C)$ in H_P . In the first, we show that the size of $\cup_{i=1}^{n'} E_i$ is equal to $s(H_R)$. Hence Lemma 2.6 implies that E is a obvious stopping set.

Lemma 2.3 implies that for each row-set B_i with $B_i \cap E \neq \emptyset$, E_i is a stopping set in H_R . Hence $\cup_{i=1}^{n'} E_i$ is a stopping set in H_R . Thus $|\cup_{i=1}^{n'} E_i| \geq s(H_R)$. The size of $\cup_{i=1}^{n'} E_i$ is equal to the number of column-sets B'_j satisfying $B'_j \cap E \neq \emptyset$. Hence the number of column-sets B'_j for which $B'_j \cap E \neq \emptyset$, is at least $s(H_R)$. The size of $B'_j \cap E$ is equal to the size of E'_j . Theorem 2.3 implies that for each column-set B'_j with $B'_j \cap E \neq \emptyset$, the set E'_j is a stopping set in H_C . Hence for each B'_j satisfying $B'_j \cap E \neq \emptyset$, the size of $B'_j \cap E$ is at least $s(H_C)$. Hence $|E| \geq s(H_R)s(H_C)$. Since $|E| = s(H_R)s(H_C)$, we have $|\cup_{i=1}^{n'} E_i| = s(H_R)$.

Now suppose E is a obvious stopping set of size $s(H_R)s(H_C)$. Hence there exist two stopping sets S_R and S_C in parity-check matrices H_R and H_C , respectively, such that $E = S_{RC}$. It follows from $|S_R| \geq s(H_R)$, $|S_C| \geq s(H_C)$ and $|S_{RC}| = s(H_R)s(H_C)$ that $|S_R| = s(H_R)$ and $|S_C| = s(H_C)$. Therefore the number stopping set of size $s(H_R)s(H_C)$ in H_P is $S_{s(H_R)}S_{s(H_C)}$. \square

Example 2.9. Let R be the linear $[3, 2, 2]$ code and let C be the $[8, 4, 4]$ Reed-Muller code with parity-check matrix H_C given in (2.3). Then we have

$$s(H_R) = 2, \quad s(H_C) = 3,$$

$$S_2(H_R) = 3, \quad S_3(H_C) = 2$$

Hence Theorem 2.8 implies that $S_6(H_P) = 6$.

3. Minimum Stopping Sets of SPC Product Codes

Let C_i , $1 \leq i \leq r$, be a binary linear code with parameters $[n_i, k_i, d_i]$. Then the corresponding r -dimensional product code can be constructed in the following way [7]. The data to be transmitted are arranged in a hypercube of dimension r with the length in each dimension defined by $\{k_1, k_2, \dots, k_r\}$. The i th dimension is encoded with C_i , and this is repeated for all $i = 1, 2, \dots, r$ dimensions. The resulting r -dimensional product code has length $\prod_{i=1}^r n_i$, minimum distance $\prod_{i=1}^r d_i$ and rate $\prod_{i=1}^r R_i$, where R_i is the rate of C_i . If component codes C_i , $1 \leq i \leq r$, are the SPC codes with same length n , then the r -dimensional product code has the length n^r and the minimum distance 2^r and the rate $(\frac{n-1}{n})^r$. This code is called an r -dimensional SPC product code. Product codes based on SPC codes have been introduced by Battail [1].

By recursive construction, we constructed a parity-check matrix H_{SPC}^r for the r -dimensional SPC product code. Let $H_{SPC} = [11 \cdots 1]$ be the parity-check matrix of the SPC code. The 1-dimensional SPC product code is the SPC code. Thus, $H_{SPC}^1 = H_{SPC}$. Using the construction (2.1), the matrix

$$(3.1) \quad H_{SPC}^r = \begin{pmatrix} H_{SPC}^{(r-1)} \otimes I_n \\ I_n \otimes H_{SPC} \end{pmatrix}$$

is a parity-check matrix of the r -dimensional SPC product code.

In the following theorem, the stopping distance of H_{SPC}^r and the number of minimum stopping sets in H_{SPC}^r is computed.

Theorem 3.1. The stopping distance of parity-check matrix H_{SPC}^r of the r -dimensional SPC product code is 2^r and the number of minimum stopping sets in H_{SPC}^r is equal to $\binom{n}{2}^r$.

Proof. The proof is by induction on r . The 1-dimensional SPC product code is the SPC code of length n . The stopping distance of H_{SPC}^1 is two and the number of stopping sets of size two is $\binom{n}{2}$. Thus the theorem holds for $r = 1$.

We now assume that the theorem holds for dimension $(r - 1)$. Then $s(H_{SPC}^{(r-1)}) = 2^{(r-1)}$ and $S_{2^{(r-1)}}(H_{SPC}^{(r-1)}) = \binom{n}{2}^{(r-1)}$. Lemma 2.7 implies that $s(H_{SPC}^r) = s(H_{SPC}^{(r-1)})s(H_{SPC}^1) = 2^r$. Using Theorem 2.8, we have

$$\begin{aligned} S_{2^r}(H_{SPC}^r) &= S_{2^{(r-1)}}(H_{SPC}^{(r-1)})S_2(H_{SPC}^1) \\ &= \binom{n}{2}^{(r-1)} \times \binom{n}{2} = \binom{n}{2}^r. \end{aligned}$$

□

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