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$\mathcal{B}$-PARTITIONS, DETERMINANT AND PERMANENT OF GRAPHS

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Abstract. Let G be a graph (directed or undirected) having k number of blocks $B_1, B_2, \dots, B_k$. A $\mathcal{B}$-partition of G is a partition consists of k vertex-disjoint subgraph $(\hat{B}_1, \hat{B}_1, \dots, \hat{B}_k)$ such that $\hat{B}_i$ is an induced subgraph of B_i for $i = 1, 2, \dots, k$. The terms $\prod_{i=1}^{k} \det(\hat{B}_i)$, $\prod_{i=1}^{k} \operatorname{per}(\hat{B}_i)$ represent the det-summands and the per-summands, respectively, corresponding to the $\mathcal{B}$-partition $(\hat{B}_1, \hat{B}_1, \dots, \hat{B}_k)$. The determinant (permanent) of a graph having no loops on its cut-vertices is equal to the summation of the det-summands (per-summands), corresponding to all possible $\mathcal{B}$-partitions. In this paper, we calculate the determinant and the permanent of classes of graphs such as block graph, block graph with negatives cliques, signed unicyclic graph, mixed complete graph, negative mixed complete graph, and star mixed block graphs.

1. Introduction

A simple graph G consists of a finite set of vertices $V(G)$ and a set of edges $E(G)$ consisting of distinct, unordered pairs of vertices. Thus, (i, j) or (j, i) represents an edge between the vertices $i, j \in V(G)$, and i, j are called adjacent vertices. If $E(G)$ consists of the ordered pairs of vertices, then G is called a directed graph or digraph. In this paper, most of the study is on the simple graphs, thus we use the term 'graph' for the simple graphs. A signed graph is a graph equipped with a weight function $f : E(G) \to \{-1, 0, 1\}$. Thus, the signed graph may have positive, negative edges with weights 1, -1, respectively. Let G be a signed graph on n vertices. The adjacency matrix $A = (a_{ij})$ of order n associated with G is defined by

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } i, j \text{ are connected with a positive edge,} \\ -1 & \text{if the vertices } i, j \text{ are connected with a negative edge,} \\ 0 & \text{if the vertices } i, j \text{ are not connected,} \end{cases}$$

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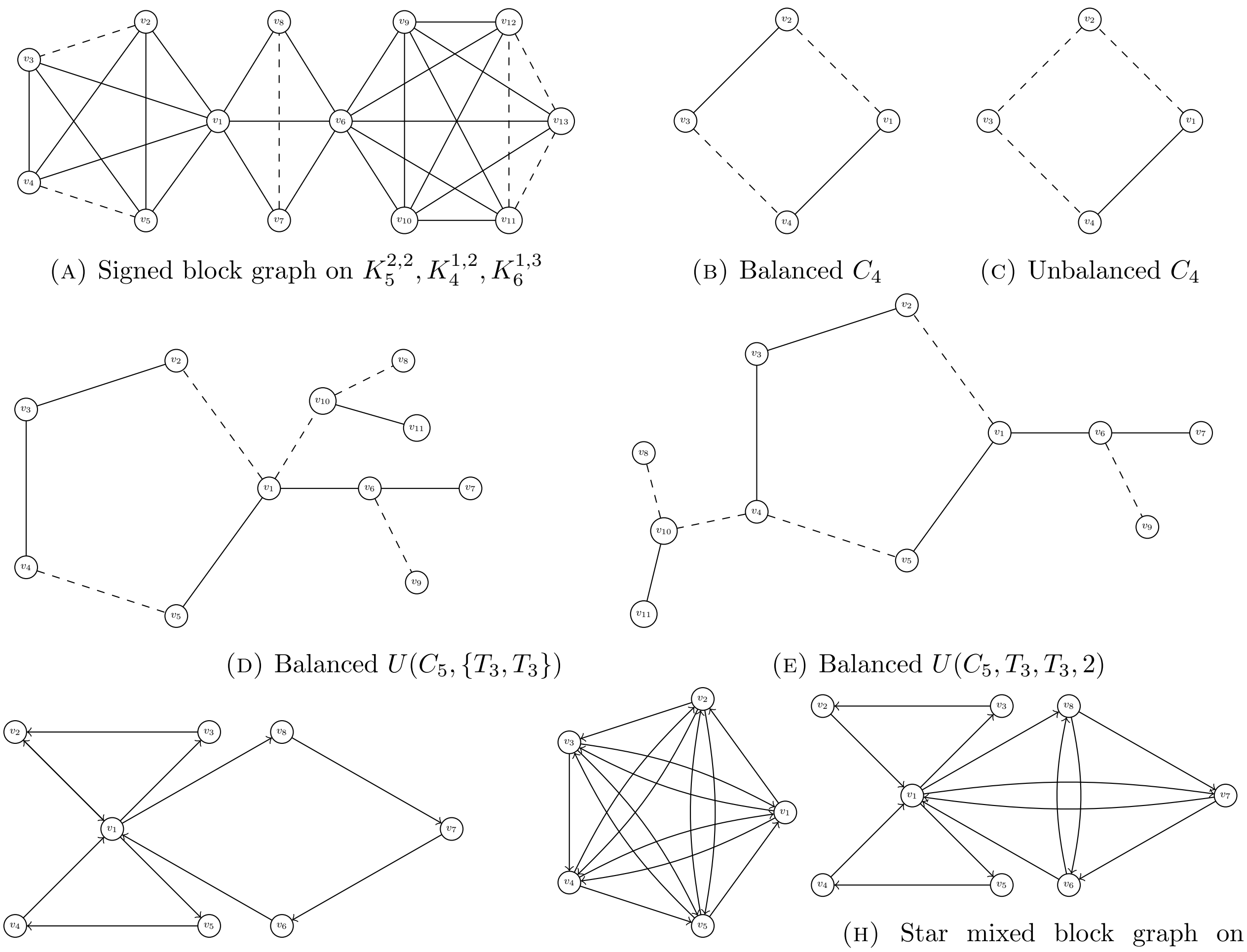
where $1 \leq i, j \leq n$. Signed graph G has an underlying graph $|G|$, in which all the negative edges are replaced by the positive edges. The corresponding adjacency matrix of $|G|$ is denoted by $|A|$. By the determinant and the permanent of a graph we mean the determinant and the permanent of its adjacency matrix.

A complete signed graph is a signed graph where each distinct pair of vertices is connected by a positive or negative edge. A signed clique in signed graph G is an induced subgraph which is a complete signed graph. When each edge of a clique is negative we call it a negative clique. Similarly, if each edge of a clique is positive, then we call it a positive clique. We denote a complete graph on n vertices, having each edge positive, by K_n. A complete graph on n vertices with the arbitrary nonzero weights on the edges is denoted by wK_n. By $K_n^{m,r}$, we denote a complete signed graph on n vertices, having m vertex-disjoint negative cliques of size r, and all the other edges positive except those are in the negative cliques [10].



(A) Signed block graph on $K_5^{2,2}, K_4^{1,2}, K_6^{1,3}$ (B) Balanced C_4 (C) Unbalanced C_4

(D) Balanced $U(C_5, \{T_3, T_3\})$ (E) Balanced $U(C_5, T_3, T_3, 2)$

(F) Star Cactoid graph on dC_5, dC_4, dC_6 (G) mK_5 (H) Star mixed block graph on mK_3, mK_3, mK_4

Figure 1. Examples: the dark line shows positive edge (weight +1), the dotted line shows negative edge (weight -1).

A path of length k between the two vertices v_1 and v_k is a sequence of distinct vertices $v_1, v_2, \ldots, v_{k-1}, v_k$, such that, $(v_i, v_{i+1}) \in E(G)$ for all $i = 1, 2, \ldots, k-1$. If $v_1 = v_k$, then the path is called a cycle. If G is a digraph, then we consider a path to be a sequence of distinct vertices $v_1, v_2, \ldots, v_{k-1}, v_k$, such that, either

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(v_i, v_{i+1}) or $(v_{i+1}, v_i) \in E(G)$ for all $i = 1, 2, \dots, k-1$. We call G to be connected if there exists a path between any two distinct vertices. A component of G is a maximally connected subgraph of G. A cut-vertex of G is a vertex whose removal results in the increase of the number of components in G. A block is a maximally connected subgraph of G that has no cut-vertex [3]. Note that, if G is a connected graph having no cut-vertex, then G itself is a block. A block having only one cut-vertex of G is called its pendant block. When each block of signed graph G is a complete signed graph, then we call it a signed block graph. We also considered the signed block graphs in which each block can have vertex-disjoint negative cliques having the same number of vertices, and edges connected to cut-vertices are positive, see Figure 1(A).

In addition to the above graphs, the weighted signed graphs are also considered, that is, the edges can have arbitrary weights. Though the meaning of weighted signed graphs and weighted graphs is apparently same, still we will use the word weighted signed graphs just to highlight the importance of the signs of the edges. Given a weighted signed graph G, a cycle is called a balanced cycle if the product of its edge weights is positive, otherwise, it is called an unbalanced cycle [4]. In other words, a balanced cycle has an even number of edges with negative weights. Figure 1(B) and Figure 1(C) are the examples of balanced and unbalanced cycles, respectively. A weighted signed graph G is called balanced graph when all the cycles in G are balanced [7]. In particular, for a complete signed graph, if all the triangles (cycles of length 3) are balanced, then it is a balanced graph [6]. The following theorem provides the spectral criterion for a weighted signed graph G to be balanced.

Theorem 1.1. [1] *A weighted signed graph G is balanced if and only if the eigenvalues of G and $|G|$ are the same.*

A signed unicyclic graph is a connected signed graph in which the number of edges equals to the number of vertices. Thus, a signed unicyclic graph is either a cycle or a cycle with trees attached to the vertices of the cycle. If the cycle is balanced, then the signed unicyclic graph is balanced, otherwise, unbalanced. A signed tree graph having m vertices is denoted by T_m. Then $U(C_n, \{T_{m_1}, T_{m_2}, \dots, T_{m_k}\})$ denotes a signed unicyclic graph having a signed cycle C_n and k signed trees $T_{m_1}, T_{m_2}, \dots, T_{m_k}$ such that the root of $T_{m_i}, i = 1, 2, \dots, k$, is linked to a fix vertex of C_n. An example of a balanced $U(C_n, \{T_{m_1}, T_{m_2}, \dots, T_{m_k}\})$ is given in Figure 1(D). By $U(C_n, T_{m_1}, T_{m_2}, l)$, we denote a unicyclic graph having a signed cycle C_n, and the roots of trees T_{m_1}, T_{m_2} are attached to two vertices v_1 and v_2 of C_n, respectively, at a distance l. An example of a balanced $U(C_n, T_{m_1}, T_{m_2}, l)$ is given in Figure 1(E).

A directed cycle dC_n is a graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$ and the edge set $E = \{(v_i, v_{i+1})\} \cup (v_n, v_1)$, $i = 1, 2, \dots, n-1$. If all the blocks of a graph are directed cycles, then the graph is called a cactoid graph. We consider a modified version of cactoid graph in which the edges can have arbitrary directions and signs. For example, see Figure 1(F). Adding all the possible arcs (directed edges) between any nonadjacent vertices of the cycle dC_n $(n > 3)$ we get a mixed complete graph mK_n, see Figure 1(G) [12]. A mixed star block graph G is a graph in which complete mixed graph are connected by one cut-vertex. An example of mixed star block graph is shown in Figure 1(H).

Any square matrix $A = (a_{ij})$ can be represented by a weighted digraph, wdG, in which an edge from vertex i to vertex j is having weight a_{ij}. A cycle cover L of wdG is a collection of the vertex-disjoint directed

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cycles that cover all the vertices. The weight $w(L)$ of cycle cover L is the product of the weights of the edges in each directed cycle. Then

$$\det(A) = (-1)^n \sum_L (-1)^{c(L)} w(L), \tag{1.1}$$

$$\operatorname{per}(A) = \sum_L w(L), \tag{1.2}$$

where $c(L)$ is the number of cycles in L, and the summation is over all the cycle covers.

The determinant of K_n is equal to $(-1)^{n-1}(n-1)$ and the permanent of K_n is given by

$$\operatorname{per}(K_n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

In this paper, we use the $\mathcal{B}$-partitions to calculate the determinant and the permanent of the above graphs. The paper is organized as the follows. We give some preliminary results on the permanent and the determinant of the weighted signed block graph in Section 2. In Section 3, the $\mathcal{B}$-partitions are used to calculate the determinant and the permanent of the block graphs. In Subsection 3.1, we calculate the determinant of a block graph having negative vertex-disjoint cliques. We find the determinant and the permanent of signed unicyclic graphs in Section 4. In Section 5, first, we find the eigenvalues of the mixed complete graph and the negative mixed complete graph, then we give their determinant expressions. Finally, we calculate the determinant of mixed star block graph as well as the determinant of the negative mixed star block graph.

2. Preliminary results

We give some preliminary results on the determinant and the permanent of balanced and unbalanced signed graphs.

Theorem 2.1. *In a weighted signed block graph G, if all the triangles are balanced, then G and $|G|$ have the same determinant.*

Proof. Each block of a weighted signed block graph G is a complete graph. As all the triangles are balanced, every block is a balanced graph [7]. Which implies that all the cycles in all the blocks of G are balanced. There cannot be any common cycle between any two blocks, thus all the cycles of G are balanced, hence G is balanced. By Theorem 1.1 G and $|G|$ have the same eigenvalues, hence G and $|G|$ have the same determinant. □

Theorem 2.2. *If a weighted signed graph G is balanced, then G and $|G|$ have the same permanent.*

Proof. A balanced graph can be partitioned into two vertex sets such that all the edges between the vertices of the same set are positive while all the edges between the vertices of different sets are negative [7]. Let X, Y be the two such sets for balanced graph G. Let S be the diagonal matrix, whose diagonal elements corresponding to the vertices in X are 1 while the elements corresponding to the vertices in Y are -1. Then $|A| = SAS$. Hence, $\operatorname{per}(|A|)=\operatorname{per}(A)(\pm 1)^2=\operatorname{per}(A)$. □

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Theorem 2.3. *Let G be a weighted signed block graph. If all the triangles in G are balanced, then G and $|G|$ have the same permanent.*

Proof. Using the proof of Theorem 2.1, if all the triangles in signed block graph G are balanced, then G is balanced. Now the theorem directly follows using Theorem 2.2. □

3. $\mathcal{B}$-partitions, determinant and permanent of block graphs

We first state a theorem for the determinant of simple block graphs given in [3]. In the theorem, the conditions on k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ can induce $\mathcal{B}$-partitions and vice versa.

Theorem 3.1. [3] *Given an unweighted block graph G of order n having blocks $B_1, B_2, \ldots, B_k$ and the adjacency matrix A,*

$$\det(A) = (-1)^{n-k} \sum \prod_{i=1}^{k} (\alpha_i - 1), \tag{3.1}$$

where the summation is over all k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of nonnegative integers satisfying the following conditions:

1. $\sum_{i=1}^{k} \alpha_i = n$;
2. *for any nonempty set $S \subseteq \{1, 2, \ldots, k\}$*

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|,$$

where G_S denotes the subgraph of G induced by the blocks B_i, $i \in S$.

Note that, under the same conditions on k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, Equation (3.1) of Theorem 3.1 can be written as

$$\det(A) = \sum \prod_{i=1}^{k} \det(K_{\alpha_i}), \tag{3.2}$$

assuming that $\det(K_0) = 1$.

Let wG be a weighted digraph having no loops on the cut-vertices. In [11, Corollary 5.1], the combinatorial expressions for the determinant and the permanent of wG are given in terms of the determinant and the permanent of subdigraphs of the blocks, respectively. The following lemma gives the determinant and the permanent of wG.

Lemma 3.2. *Let wG be a weighted digraph having no loops on its cut-vertices. Let $B_1, B_2, \ldots, B_k$ be the blocks in it. Then*

$$\det(wG) = \sum \prod_{i=1}^{k} \det(\hat{B}_i),$$

$$\operatorname{per}(wG) = \sum \prod_{i=1}^{k} \operatorname{per}(\hat{B}_i),$$

where if $\hat{B}_i$ is a null graph, then $\det(\hat{B}_i) = \operatorname{per}(\hat{B}_i) = 1$. The summation is over all the possible k-combinations of induced subgraphs $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k$ such that,

(1) $\hat{B}_i \subseteq B_i$,
(2) $\bigcup_{i=1}^k V(\hat{B}_i) = V(wG)$,
(3) $V(\hat{B}_i) \bigcap V(\hat{B}_j) = \phi$, *for* $i \neq j$,

for $i, j = 1, 2, \ldots, k$.

Thus, the summation is over all the k-combinations $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k$ of induced subgraphs which partition wG. These partitions are called as $\mathcal{B}$-partitions, and the corresponding terms $\prod_{i=1}^k \det(\hat{B}_i)$, $\prod_{i=1}^k \operatorname{per}(\hat{B}_i)$ are called det-summands, per-summands, respectively. We will now prove that each k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ in Theorem 3.1 produces a unique $\mathcal{B}$-partition of any weighted graph and vice versa.

Lemma 3.3. *Let G be a graph with n vertices and k blocks. Let $B_1, B_2, \ldots, B_k$ be its blocks having $b_1, b_2, \ldots, b_k$, number of vertices, respectively. Then each $\mathcal{B}$-partition produces a unique k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of nonnegative integers satisfying the following conditions:*

(1) $\sum_{i=1}^k \alpha_i = n$;
(2) *for any nonempty set* $S \subseteq \{1, 2, \ldots, k\}$

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|,$$

where G_S denotes the subgraph of G induced by the blocks B_i, $i \in S$.

Proof. By Lemma 3.2, the determinant and the permanent of G are equal to

$$\sum \prod_{i=1}^k \det(\hat{B}_i), \ \sum \prod_{i=1}^k \operatorname{per}(\hat{B}_i),$$

respectively, where $\hat{B}_i \subseteq B_i$, $i = 1, 2, \ldots, k$. The summations are over all the $\mathcal{B}$-partitions of G.

The vertex-disjoint induced subgraphs $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k$ create a $\mathcal{B}$-partition of G. Thus $\sum_{i=1}^k |V(\hat{B}_i)| = n$, and for any nonempty set $S \subseteq \{1, 2, \ldots, k\}$,

$$\sum_{i \in S} |V(\hat{B}_i)| \leq |V(G_S)|,$$

where G_S denotes the subgraph of G induced by the blocks B_i, $i \in S$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the number of vertices in a given $\mathcal{B}$-partition $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k$, respectively. Thus k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ resulted from $\mathcal{B}$-partitions of G satisfy both the conditions of the theorem.

Conversely, consider a k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying both conditions of the theorem. We will prove by induction that each such k-tuple corresponds to a unique $\mathcal{B}$-partition of G.

If G has only one block B_1 of order b_1, then the only possible choice for 1-tuple is $\alpha_1 = b_1$. Clearly, α_1 corresponds to a $\mathcal{B}$-partition which consists of B_1 only. Let G has two blocks B_1 and B_2 of order b_1 and b_2, respectively, and the cut-vertex v. The possible 2-tuples are $(\alpha_1 = b_1, \ \alpha_2 = b_2 - 1)$ and $(\alpha_1 = b_1 - 1, \ \alpha_2 = b_2)$. Both the 2-tuple induce possible two $\mathcal{B}$-partitions in G. One $\mathcal{B}$-partition consists of the induced subgraphs $B_1, B_2 \setminus v$. Another $\mathcal{B}$-partition consists of the induced subgraphs $B_1 \setminus v, B_2$.

Now we discuss the proof of G consisting of three blocks, which will clarify the reasoning for the general case. For the time being, let us denote the graph having k blocks by G_k. Let the blocks be $B_1, B_2, \ldots, B_k$ of order $b_1, b_2, \ldots, b_k$, respectively. The formation of a G_k can be seen as the k-step process. At any

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intermediate i-th step a block B_i is added to G_{i-1} and then B_i becomes a pendant block for G_i. In G_3, the block B_3 can occur in two ways.

(1) Let B_3 be added to a noncut-vertex of G_2. Without loss of generality, let B_3 get attached to a non-cut-vertex of B_2 in G_2. In the resulting G_3, let v_1 be the cut-vertex in B_1, B_2, and v_2 be the cut-vertex in B_2, B_3. Choices for 3-tuple $(\alpha_1, \alpha_2, \alpha_3)$ are the following:
   (a) $\alpha_1 = b_1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3 - 1$;
   (b) $\alpha_1 = b_1,\ \alpha_2 = b_2 - 2,\ \alpha_3 = b_3$;
   (c) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2,\ \alpha_3 = b_3 - 1$;
   (d) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3$.
   Note that, in this case, each 2-tuple of G_2 give rise to two 3-tuple in G_3 where α_1 is unchanged. Clearly, all the tuples in G_3 can induce the following possible $\mathcal{B}$-partitions.
   (a) $B_1,\ B_2 \setminus v_1,\ B_3 \setminus v_2$;
   (b) $B_1,\ B_2 \setminus (v_1, v_2),\ B_3$;
   (c) $B_1 \setminus v_1,\ B_2,\ B_3 \setminus v_2$;
   (d) $B_1 \setminus v_1,\ B_2 \setminus v_2,\ B_3$.
(2) Let B_3 be added to cut-vertex v of G_2. Choices for 3-tuple $(\alpha_1, \alpha_2, \alpha_3)$ are the following:
   (a) $\alpha_1 = b_1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3 - 1$;
   (b) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2,\ \alpha_3 = b_3 - 1$;
   (c) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3$.
   Here, each 2-tuple of G_2 give rise to a 3-tuple of G_3 where α_1, α_2 are unchanged and $\alpha_3 = b_3 - 1$. Beside these, there is one more 3-tuple where $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2 - 1, \alpha_3 = b_3$. Clearly, all the tuples in G_3 can induce the following possible $\mathcal{B}$-partitions.
   (a) $B_1,\ B_2 \setminus v,\ B_3 \setminus v$;
   (b) $B_1 \setminus v,\ B_2,\ B_3 \setminus v$;
   (c) $B_1 \setminus v,\ B_2 \setminus v,\ B_3$.

Now, let us assume that all the possible k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ in G_k can induce all the possible $\mathcal{B}$-partitions in it. We need to prove that all the possible $(k+1)$-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1})$ in G_{k+1} can induce its all the possible $\mathcal{B}$-partitions. In G_{k+1} the block B_{k+1} can occur in two ways.

(1) Let B_{k+1} be added to a non cut-vertex of G_k. Each k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of G_k give rise to two $(k+1)$-tuple of G_{k+1}, where $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ are unchanged. In one such tuple α_k is also unchanged and $\alpha_{k+1} = b_{k+1} - 1$. In the other tuple α_k is one less than the value it had earlier and $\alpha_{k+1} = b_{k+1}$. Thus, the $(k+1)$-tuples can induce all the $\mathcal{B}$-partitions in G_{k+1}.
(2) Let B_{k+1} be added to a cut-vertex v of G_k. Each k-tuple of G_k give rise to one $(k+1)$-tuple of G_{k+1}, where $\alpha_{k+1} = b_{k+1} - 1$. Beside these, there are also $(k+1)$-tuples where $\alpha_{k+1} = b_{k+1}$, along with k-tuples of $(G_k \setminus v)$. Clearly, all the tuples in G_{k+1} can induce its $\mathcal{B}$-partitions.

Hence, there is one to one correspondence between the $\mathcal{B}$-partitions and the k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_3)$. □

Now we give a formula for the permanent of balanced signed block graphs.

Theorem 3.4. *Let G be a balanced signed block graph with n vertices and having all the edges of weight 1. Let $B_1, B_2, \ldots, B_k$ be its blocks. Let A be the adjacency matrix of G. Then*

$$\operatorname{per}(A) = \sum \prod_{i=1}^{k} \alpha_i! \sum_{j=0}^{\alpha_i} \frac{(-1)^j}{j!}, \tag{3.3}$$

where the summation is over all k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of nonnegative integers satisfying the following conditions:

1. $\sum_{i=1}^{k} \alpha_i = n$;
2. *for any nonempty set $S \subseteq \{1, 2, \ldots, k\}$*

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|,$$

where G_S denotes the subgraph of G induced by the blocks B_i, $i \in S$.

Proof. The proof follows by Lemma 3.2, Lemma 3.3 and the fact that

$$\operatorname{per}(K_{\alpha_i}) = \alpha_i! \sum_{j=0}^{\alpha_i} \frac{(-1)^j}{j!}.$$

□

3.1. **Block graph with negative cliques.** First, we give the determinant of a complete graph with negative cliques, $K_n^{m,r}$. Subsequently, the determinant of the block graph with negative cliques is given.

Lemma 3.5. [10, Corollary 3.6] *The determinant of $A(K_n^{m,r})$ is given by*

$$(1-2r)^{m-1}(-1)^{n-mr-1}\Big(n(1-2r) + 2r\Big(1 + m(r-1)\Big) - 1\Big).$$

Theorem 3.6. *Let G be a signed block graph of order n having k blocks $B_1, B_2, \ldots, B_k$. Let all the edges connecting cut-vertices be positive. Let B_i has m_i number of vertex-disjoint negative cliques each of size r_i, such that $0 \leq m_i r_i \leq (n_i - 1)$, $i = 1, 2, \ldots, k$. Then*

$$\det(G) = (-1)^{n-k} \sum \prod_{i=1}^{k} (1-2r_i)^{m_i-1}(-1)^{-m_i r_i}\Big(\alpha_i(1-2r_i) + 2r_i\Big(1 + m_i(r_i - 1)\Big) - 1\Big), \tag{3.4}$$

where the summation is over all k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of nonnegative integers satisfying the following conditions:

1. $\sum_{i=1}^{k} \alpha_i = n$;
2. *for any nonempty set $S \subseteq \{1, 2, \ldots, k\}$*

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|,$$

where G_S denotes the subgraph of G induced by the blocks B_i, $i \in S$.

Proof. The result directly follows by Lemma 3.2, Lemma 3.3 and Lemma 3.5. □

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4. Determinant and permanent of signed unicyclic graphs

Let U be a unicyclic graph which contains a signed cycle C_n as a subgraph with the vertices $v_1, v_2, \dots, v_n$. Assume that the vertex v_i is linked with m_i number of signed trees say $T^i_1, T^i_2, \dots, T^i_{m_i}$, such that the root vertex of each $T^i_j, j = 1, 2, \dots, m_i$ is linked with v_i by an edge. Note that the vertex v_i now becomes a cut-vertex. As tree is acyclic graph, determinant and permanent of any signed tree is equal to the determinant and the permanent of its underlying tree with positive edges. Let $\{T^i_1, T^i_2, \dots, T^i_{m_i}\}$ denotes the subgraph of U induced by the trees $T^i_j, j = 1, \dots, m_i$. Let $U \setminus \{T^i_1, T^i_2, \dots, T^i_{m_i}\}$ denotes the induced subgraph of U after $\{T^i_1, T^i_2, \dots, T^i_{m_i}\}$ is removed from U, and $\{T^i_1, T^i_2, \dots, T^i_{m_i}, v_i\}$ denotes the subgraph of U induced by trees $T^i_1, T^i_2, \dots, T^i_{m_i}$ and vertex v_i. In [11], the Lemma 2.3 and the Corollary 2.4, can be re-written for the determinant and the permanent, respectively for the graphs with no loop on the cut-vertices.

Lemma 4.1. *Let G be a digraph with at least one cut-vertex. Let H be a nonempty subdigraph of G having cut-vertex v, such that $H \setminus v$ is a union of connected components. Then*

$$\det(G) = \det(H) \times \det(G \setminus H) + \det(H \setminus v) \times \det\Big(G \setminus (H \setminus v)\Big). \tag{4.1}$$

Corollary 4.2. *Let G be a digraph with at least one cut-vertex. Let H be a nonempty subdigraph of G having cut-vertex v, such that $H \setminus v$ is a union of connected components. Then*

$$\operatorname{per}(G) = \operatorname{per}(H) \times \operatorname{per}(G \setminus H) + \operatorname{per}(H \setminus v) \times \operatorname{per}\Big(G \setminus (H \setminus v)\Big). \tag{4.2}$$

Using Lemma 4.1 on U at v_i we get

$$\begin{aligned} \det(U) &= \det\Big(U \setminus \{T^i_1, T^i_2, \dots, T^i_{m_i}\}\Big) \det\Big(\{T^i_1, T^i_2, \dots, T^i_{m_i}\}\Big) \\ &\quad + \det\Big(U \setminus \{T^i_1, T^i_2, \dots, T^i_{m_i}, v_i\}\Big) \det\Big(\{T^i_1, T^i_2, \dots, T^i_{m_i}, v_i\}\Big). \end{aligned} \tag{4.3}$$

Using Corollary 4.2 on U at v_i we get

$$\begin{aligned} \operatorname{per}(U) &= \operatorname{per}\Big(U \setminus \{T^i_1, T^i_2, \dots, T^i_{m_i}\}\Big) \operatorname{per}\Big(\{T^i_1, T^i_2, \dots, T^i_{m_i}\}\Big) \\ &\quad + \operatorname{per}\Big(U \setminus \{T^i_1, T^i_2, \dots, T^i_{m_i}, v_i\}\Big) \operatorname{per}\Big(\{T^i_1, T^i_2, \dots, T^i_{m_i}, v_i\}\Big). \end{aligned} \tag{4.4}$$

Now we have the following results.

Theorem 4.3. *Consider a unicyclic signed graph $U(C_n, T_m)$ where a signed tree T_m is linked with the signed cycle C_n by an edge between the root vertex of T_m and a vertex v of C_n. Then*

$$\det\Big(U(C_n, T_m)\Big) = \begin{cases} 0, & \text{if } n \text{ is even and } T_m \text{ has no perfect matching} \\ (-1)^{\frac{m}{2}}\Big(-2\delta + 2(-1)^{\frac{n}{2}}\Big), & \text{if } n \text{ is even and } T_m \text{ has a perfect matching} \\ (-1)^{\frac{m+n}{2}}, & \text{if } n \text{ is odd and } \{T_m, v\} \text{ has a perfect matching} \\ 2\delta(-1)^{\frac{m}{2}}, & \text{if } n \text{ is odd and } T_m \text{ has a perfect matching} \end{cases}$$

where $\delta = 1$ if C_n is balanced, otherwise, $\delta = -1$.

Proof. Let the tree T_m be attached to C_n via an edge between the vertex u_1 of T_m and the vertex v of C_n. Using Lemma 4.1 the determinant of $U(C_n, T_m)$ can be written as

$$\begin{aligned} \det\Big(U(C_n, T_m)\Big) &= \det(C_n) \times \det(T_m) + \det(C_n \setminus v) \times \det(\{T_m, v\}) \\ &= \det(C_n) \times \det(T_m) + \det(P_{n-1}) \times \det(\{T_m, v\}), \end{aligned} \tag{4.5}$$

where $C_n \setminus v$ is the subgraph in which the vertex v is removed from C_n and hence it becomes P_{n-1}. As signed tree without a perfect matching has determinant zero, using [10, Corollary 2.3], the determinant of signed cycle C_n having weight $\delta \in \{-1, 1\}$ is given by

$$\det(C_n) = \begin{cases} 2 - 2\delta & \text{if } n \text{ is even and even multiple of 2} \\ -2 - 2\delta & \text{if } n \text{ is even and odd multiple of 2} \\ 2\delta & \text{if } n \text{ is odd} \end{cases}$$

Now we consider the following cases.

Case I n is even and T_m has no perfect matching: As in this case, $\det(T_m) = 0$, $\det(P_{n-1}) = 0$. Using Equation (4.5) $\det(U(C_n, T_m)) = 0$.

Case II n is even and T_m has a perfect matching: Consider $n = 2k, m = 2k'$, where $k \geq 2$ and $k' \geq 1$ are positive integers. As $\det(P_{n-1}) = 0$, using Equation (4.5)

$$\det(U(C_n, T_m)) = \det(C_n) \times \det(T_m) = (-2\delta + 2(-1)^k)(-1)^{k'},$$

where for balanced C_n, $\delta = 1$ and for unbalanced C_n, $\delta = -1$.

Case III if n is odd and $\{T_m, v\}$ has a perfect matching: If m is odd, then $\det(T_m) = 0$. Using Equation (4.5)

$$\det(U(C_n, T_m)) = \det(P_{n-1}) \times \det(\{T_m, v\}).$$

If $\{T_m, v\}$ has no perfect matching, then $\det(U(C_n, T_m)) = 0$. Otherwise,

$$\det(U(C_n, T_m)) = (-1)^{\frac{n-1}{2}}(-1)^{\frac{m+1}{2}} = (-1)^{\frac{n+m}{2}}.$$

Case IV n is odd and T_m has a perfect matching: In this case $m + 1$ is an odd number, so $\det(\{T_m, v\}) = 0$. Thus, using Equation (4.5)

$$\det(U(C_n, T_m)) = \det(C_n) \times \det(T_m) = 2\delta(-1)^{\frac{m}{2}},$$

where for the balanced C_n, $\delta = 1$ and for the unbalanced C_n, $\delta = -1$.

□

Corollary 4.4. *Consider a unicyclic signed graph $U(C_n, T_m)$ as in Theorem 4.3. Then*

$$\mathrm{per}\Big(U(C_n, T_m)\Big) = \begin{cases} 0, & \text{if } n \text{ is even and } T_m \text{ has no perfect matching} \\ -2\delta + 2, & \text{if } n \text{ is even and } T_m \text{ has a perfect matching} \\ 1, & \text{if } n \text{ is odd and } \{T_m, v\} \text{ has a perfect matching} \\ 2\delta, & \text{if } n \text{ is odd and } T_m \text{ has a perfect matching} \end{cases}$$

where $\delta = 1$ if C_n is balanced, otherwise, $\delta = -1$.

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Proof. Using Equation (1.2)

$$\mathrm{per}(C_n) = \begin{cases} 2-2\delta & \text{if } n \text{ is even} \\ 2\delta & \text{if } n \text{ is odd.} \end{cases}$$

Rest of the steps are similar to Theorem 4.3. □

Theorem 4.5. *Let $U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})$ denotes a unicyclic graph having a signed cycle C_n and k signed trees $T_{m_1}, T_{m_2}, \ldots, T_{m_k}$. Assume that root of each T_{m_i} is linked with vertex v of C_n by an edge for all $i = 1, 2, \ldots, k$. Then*

$$\begin{aligned}\det\Big(U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})\Big) &= \det(C_n)\prod_{i=1}^{k}\det(T_{m_i}) \\ &\quad + \det(P_{n-1})\sum_{i=1}^{k}\Big(\det(\{T_{m_i}, v\})\prod_{j=1, j\neq i}^{k}\det(T_{m_j})\Big).\end{aligned}$$

Proof. Using Equation (4.3) observe that

$$\begin{aligned}\det\Big(U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})\Big) &= \det\Big(U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}) \setminus \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}\Big) \\ &\quad \times \det(\{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}) \\ &\quad + \det\Big(U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}) \setminus \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}, v\}\Big) \\ &\quad \times \det(\{T_{m_1}, T_{m_2}, \ldots, T_{m_k}, v\}),\end{aligned}$$

where $U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}) \setminus \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\} = C_n$. As $\{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}$ is the induced subgraph of the unicyclic graph having k connected components $T_{m_i}, i = 1, \ldots, k$,

$$\det\Big(\{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}\Big) = \prod_{i=1}^{k}\det(T_{m_i}).$$

Next, $U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}) \setminus \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}, v\} = P_{n-1}$. The only thing that is left to know is $\det(\{T_{m_1}, T_{m_2}, \ldots, T_{m_k}, v\})$. Again using Lemma 4.1 on $\{T_{m_1}, T_{m_2}, \ldots, T_{m_k}, v\}$ at v

$$\det\Big(\{T_{m_1}, T_{m_2}, \ldots, T_{m_k}, v\}\Big) = \sum_{i=1}^{k}\Big(\det(\{T_{m_i,v}\})\prod_{j=1, j\neq i}^{k}\det(T_{m_j})\Big).$$

Thus, the desired result follows. □

Corollary 4.6. *Let $U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})$ denotes a unicyclic graph as considered in Theorem 4.5. Then*

$$\begin{aligned}\mathrm{per}(U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})) &= \mathrm{per}(C_n)\prod_{i=1}^{k}\mathrm{per}(T_{m_i}) \\ &\quad + \mathrm{per}(P_{n-1})\sum_{i=1}^{k}\Big(\mathrm{per}(\{T_{m_i}, v\})\prod_{j=1, j\neq i}^{k}\mathrm{per}(T_{m_j})\Big).\end{aligned}$$

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Theorem 4.7. *Let $U(C_n, T_{m_1}, T_{m_2}, l)$ denotes a signed unicyclic graph having a signed cycle C_n and two trees T_{m_1}, T_{m_2} attached by edges to two vertices v_1 and v_2 of C_n, respectively, at a distance l. Then*

$$\det\Big(U(C_n, T_{m_1}, T_{m_2}, l)\Big) = \det\Big(U(C_n, T_{m_2})\Big)\det(T_{m_1}) + \det(\{T_{m_1}, v_1\})\det(\{T_{m_2}, v_{l+1}\})\det(P_{l-1})\det(P_{n-l-1}) + \det(\{T_{m_1}, v_1\})\det(\{T_{m_2}\})\det(P_{n-1}).$$

Proof. Using Equation (4.3) it follows that

$$\det\Big(U(C_n, T_{m_1}, T_{m_2}, l)\Big) = \det\Big(U(C_n, T_{m_1}, T_{m_2}, l) \setminus \{T_{m_1}\}\Big)\det(T_{m_1}) + \det\Big(U(C_n, T_{m_1}, T_{m_2}, l) \setminus \{T_{m_1}, v_1\}\Big)\det(\{T_{m_1}, v_1\}). \tag{4.6}$$

Note that, $\det\Big(U(C_n, T_{m_1}, T_{m_2}, l) \setminus \{T_{m_1}\}\Big) = \det\Big(U(C_n, T_{m_2})\Big)$, and $\{T_{m_1}, v_1\}$ is a tree with $m_1 + 1$ vertices. The only thing remains to figure out is $\det\Big(U(C_n, T_{m_1}, T_{m_2}, l) \setminus \{T_{m_1}, v_1\}\Big)$. Let for the time being denote $U(C_n, T_{m_1}, T_{m_2}, l)$ by U. Using Lemma 4.1 on $U \setminus \{T_{m_1}, v_1\}$ at v_2

$$\det\Big(U \setminus \{T_{m_1}, v_1\}\Big) = \det(\{T_{m_2}, v_2\})\det\Bigg(\Big(U \setminus \{T_{m_1}, v_1\}\Big) \setminus \{T_{m_2}, v_2\}\Bigg) + \det(\{T_{m_2}\})\det\Bigg(\Big(U \setminus \{T_{m_1}, v_1\}\Big) \setminus \{T_{m_2}\}\Bigg).$$

Further observe that $\Big(U \setminus \{T_{m_1}, v_1\}\Big) \setminus \{T_{m_2}, v_2\}$ is a disconnected subgraph with two connected components P_{l-1} and $P_{n-(l+1)}$, hence

$$\det\Bigg(\Big(U \setminus \{T_{m_1}, v_1\}\Big) \setminus \{T_{m_2}, v_2\}\Bigg) = \det(P_{l-1})\det(P_{n-l-1}),$$

and $\Big(U \setminus \{T_{m_1}, v_1\}\Big) \setminus \{T_{m_2}\} = P_{n-1}$. Thus, the desired result follows. □

Corollary 4.8. *Let $U(C_n, T_{m_1}, T_{m_2}, l)$ be a signed unicyclic as considered in Theorem 4.7. Then*

$$\text{per}(U(C_n, T_{m_1}, T_{m_2}, l)) = \text{per}\Big(U(C_n, T_{m_2})\Big)\text{per}(T_{m_1}) + \text{per}(\{T_{m_1}, v_1\})\text{per}(\{T_{m_2}, v_2\})\text{per}(P_{l-1})\text{per}(P_{n-l-1}) + \text{per}(\{T_{m_1}, v_1\})\text{per}(\{T_{m_2}\})\text{per}(P_{n-1}).$$

5. Mixed complete graph, mixed star block graph.

The adjacency matrix $A(mK_n)$, of mixed complete graph mK_n can be written as:

$$A(mK_n) = J_n I_n - Q_n,$$

where J_n is the all-one matrix, I_n is an identity matrix, and Q_n is the full-cycle permutation matrix of order n. Thus, the $(i, i+1)$-element of Q_n is 1, $i = 1, 2, \ldots, n-1$, the $(n, 1)$-element of Q_n is 1, and the remaining elements of Q_n are zero [2].

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The eigenvalues of Q_n are $w^i, 0 \leq i \leq n-1$, and corresponding eigenvectors are

$$v_i = [1, w^i, w^{2i}, \ldots, w^{(n-1)i}]^T,$$

$0 \leq i \leq n-1$, where w is an n-th primitive root of 1. The eigenvectors are orthogonal to each other, that is, $v_i^T v_j = 0$ for $0 \leq i, j \leq n-1$. Note that v_0 is the all-one column vector. Then the eigenvalues of $A(mK_n)$ are $\lambda_0 = n-2$ and $\lambda_i = -1 - w^i, 1 \leq i \leq n-1$.

Lemma 5.1.

$$\prod_{i=1}^{n-1}(-1-w^i) = \begin{cases} 0 & \textit{if } n \textit{ is even} \\ 1 & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. As

$$x^n - 1 = (x-1)\prod_{i=1}^{n-1}(x-w^i),$$

$$\implies \sum_{i=1}^{n} x^{n-i} = \prod_{i=1}^{n-1}(x-w^i).$$

Hence, the result follows. □

Theorem 5.2. *The determinant of $A(mK_n)$ is given by*

$$\det\Big(A(mK_n)\Big) = \begin{cases} 0 & \textit{if } n \textit{ is even} \\ (n-2) & \textit{if } n \textit{ is odd} \end{cases} \tag{5.1}$$

Proof. As the eigenvalues of $A(mK_n)$ are $\lambda_0 = n-2$ and $\lambda_i = -1 - w^i$, $1 \leq i \leq n-1$.

$$\det\Big(A(mK_n)\Big) = (n-2)\prod_{i=1}^{n-1}(-1-w^i).$$

The proof directly follows using Lemma 5.1. □

5.1. **Mixed star block graph.** A mixed block graph is a strongly connected directed graph whose blocks are mixed complete graphs. A mixed block graph having maximum one cut-vertex is called mixed star block graph, see Figure 1(C). In other words, a mixed star block graph is obtained from a star cactoid graph after adding all possible directed edges between any two nonadjacent vertices in each block. As a star cactoid graph cannot have cycle cover it is evident that it is singular. Let $mK_n \setminus v_i$ denotes an induced subgraph resulting after vertex v_i is removed from mK_n.

Lemma 5.3. *The determinant of $mK_n \setminus v_i (i = 1, 2, \ldots, n)$ is given by*

$$(-1)^n\Big(\lfloor\frac{n-2}{2}\rfloor\Big).$$

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Proof. Without loss of generality let us remove the first vertex v_1 of mK_n. The adjacency matrix of $mK_n \setminus v_1$ can be written as

$$A\Big(mK_n \setminus v_1\Big) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \ddots & 1 \\ 1 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 & 0 \end{bmatrix}.$$

In other words, $A\Big(mK_n \setminus v_1\Big)$ is a square matrix of size $n-1$ whose diagonal and sub-diagonal elements are zero and rest of the elements are 1. Let R_i denotes the i-th row of $A\Big(mK_n \setminus v_1\Big)$. In order to calculate the determinant, let us first make the following elementary row operations.

(1) $R_i = R_i - R_{i+1}$ for $i = 1, 2, \cdots, (n-2)$.
(2) Add all the resulting $n-2$ rows in 1. to $(n-1)$-th row.

The resulting matrix is

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 0 & 1 \\ 0 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

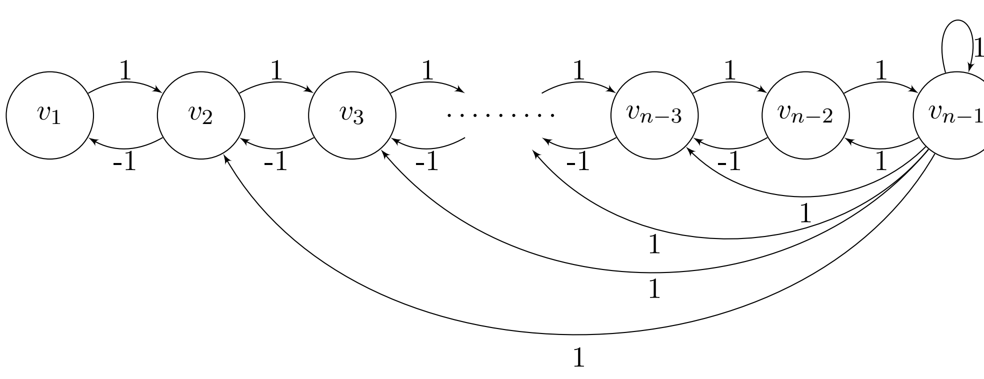

Figure 2. Digraph of the matrix $A\Big(mK_n \setminus v_1\Big)$ after the elementary operations

The digraph corresponding to the above matrix is shown in Figure 2. Using the cycle covers of the digraph we calculate the determinant.

(1) n is odd: In this case the cycle covers are the following. For $i = 1, 2, \ldots, \frac{n-3}{2}$, in a cycle cover there are directed 2-cycles, each having weight -1, on the vertices $\{v_{2j-1}, v_{2j}\}$, $j = 1, 2, \ldots, i$, and a

directed $(n-1-2i)$-cycle of weight 1 on the vertices $\{v_{n-1}, v_{2i+1}, v_{2i+2}, \cdots, v_{n-1}\}$. Hence,

$$\begin{aligned} det\Big(A\big(mK_n \setminus v_1\big)\Big) &= (-1)^{n-1}\sum_{i=1}^{\frac{n-3}{2}}(-1)^{i+1}\times(-1)^i\times 1 \\ &= \frac{3-n}{2}. \end{aligned} \tag{5.2}$$

(2) n is even: In this case the cycle covers are the following. For $i = 1, 2, \ldots, \frac{n-4}{2}$, in a cycle cover there are directed 2-cycles, each having weight -1, on the vertices $\{v_{2j-1}, v_{2j}\}$, $j = 1, 2, \ldots, i$, and a directed $(n-1-2i)$-cycle of weight 1 on the vertices $\{v_{n-1}, v_{2i+1}, v_{2i+2}, \cdots, v_{n-1}\}$. Other than these there is one more cycle cover having loop at vertex v_{n-1}, and $\frac{n-2}{2}$ directed 2-cycles on $\{v_{2i-1}, v_{2i}\}, i = 1, 2, \ldots, \frac{n-2}{2}$ each of weight -1. Hence,

$$\begin{aligned} det\Big(A\big(mK_n \setminus v_1\big)\Big) &= (-1)^{n-1}\Bigg(\sum_{i=1}^{\frac{n-4}{2}}(-1)^{i+1}\times(-1)^i\times 1 + (-1)^{1+\frac{n-2}{2}}\times(-1)^{\frac{n-2}{2}}\times 1\Bigg) \\ &= \frac{n-2}{2}. \end{aligned} \tag{5.3}$$

The result follows by Equation (5.2) and Equation (5.3). ☐

Theorem 5.4. *Let mG be mixed star block graph having k blocks $B_1, B_2, \ldots, B_k$ of order $n_1, n_2, \ldots, n_k$, respectively, then*

$$\det(mG) = \sum \det(mK_{n_i}) \prod_{j=1, j\neq i}^{k} (-1)^{n_j}\Big(\lfloor\frac{n_j-2}{2}\rfloor\Big),$$

where the summation is over all i such that n_i is odd.

Proof. Let v be the cut-vertex of mG. Using Lemma 5.3 and Lemma 3.2

$$\begin{aligned} \det(mG) &= \sum_{i=1}^{k}\det(mK_{n_i})\prod_{j=1, j\neq i}^{k}\det\Big(mK_{n_i}\setminus v\Big) \\ &= \sum_{i=1}^{k}\det(mK_{n_i})\prod_{j=1, j\neq i}^{k}(-1)^{n_j}\Big(\lfloor\frac{n_j-2}{2}\rfloor\Big). \end{aligned} \tag{5.4}$$

Using Lemma 5.1, $\det(mK_{n_i}) = 0$ for even n_i. Hence,

$$\det(mG) = \sum \det(mK_{n_i}) \prod_{j=1, j\neq i}^{k} (-1)^{n_j}\Big(\lfloor\frac{n_j-2}{2}\rfloor\Big),$$

where the summation is over all i such that n_i is odd. ☐

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5.2. **Negative mixed complete graph.** A negative directed cycle dC_n is cycle graph whose each directed edge is negative, that is, each of its edges have weight -1. A negative mixed complete graph $\overline{m}K_n$ is obtained from a negative directed cycle dC_n of length $n > 3$ by adding all the possible positive arcs between any nonadjacent vertices of the underlying cycle C_n. The adjacency matrix $A(\overline{m}K_n)$ can be written as:

$$A(\overline{m}K_n) = J_nI_n - 2Q_n - Q^{n-1},$$

where J_n is the all-one matrix, I_n is an identity matrix, and Q_n is the full-cycle permutation matrix of order n. Then the eigenvalues of $A(\overline{m}K_n)$ are $\lambda_0 = n-4$ and $\lambda_i = -1-2w^i - w^{i(n-1)}(1 \leq i \leq n-1)$, where $w = e^{\frac{2\pi\iota}{n}}$.

Lemma 5.5. *The determinant of $A(\overline{m}K_n)$ is given by*

$$\det\Big(A(\overline{m}K_n)\Big) = \begin{cases} 2(n-4)\prod_{i=1}^{\frac{(n-2)}{2}}\left(2+8\cos^2\left(\frac{2\pi i}{n}\right)+6\cos\left(\frac{2\pi i}{n}\right)\right), & \text{if } n \text{ is even} \\ (n-4)\prod_{i=1}^{\frac{(n-1)}{2}}\left(2+8\cos^2\left(\frac{2\pi i}{n}\right)+6\cos\left(\frac{2\pi i}{n}\right)\right), & \text{if } n \text{ is odd.} \end{cases} \tag{5.5}$$

Proof. For $i = 1, 2, \ldots, (n-1), w^i = \cos\left(\frac{2\pi i}{n}\right) + \iota\sin\left(\frac{2\pi i}{n}\right)$, and

$$\begin{aligned}\lambda_i &= -1-2w^i - w^{i(n-1)} \\ &= -1-2w^i-w^{-i} \\ &= -1-3\cos\left(\frac{2\pi i}{n}\right) - \iota\sin\left(\frac{2\pi i}{n}\right).\end{aligned}$$

Now, $3\cos\left(\frac{2\pi(n-i)}{n}\right) - \iota\sin\left(\frac{2\pi(n-i)}{n}\right) = 3\cos\left(\frac{2\pi i}{n}\right) + \iota\sin\left(\frac{2\pi i}{n}\right)$, if n is even, then $\lambda_{n/2} = 2$. The following are the determinant expressions for $A(\overline{m}K_n)$.

(1) n is odd:

$$\begin{aligned}\det\Big(A(\overline{m}K_n)\Big) &= (n-4)\prod_{i=1}^{\frac{(n-1)}{2}}\left(\left(-1-3\cos\left(\frac{2\pi i}{n}\right)\right)^2+\sin^2\left(\frac{2\pi i}{n}\right)\right) \\ &= (n-4)\prod_{i=1}^{\frac{(n-1)}{2}}\left(2+8\cos^2\left(\frac{2\pi i}{n}\right)+6\cos\left(\frac{2\pi i}{n}\right)\right).\end{aligned}$$

(2) n is even:

$$\det\Big(A(\overline{m}K_n)\Big) = 2(n-4)\prod_{i=1}^{\frac{(n-2)}{2}}\left(2+8\cos^2\left(\frac{2\pi i}{n}\right)+6\cos\left(\frac{2\pi i}{n}\right)\right).$$

□

5.3. **Determinant of a negative mixed star block graph.** A negative mixed block graph is a strongly connected directed graph whose blocks are negative mixed complete graphs. A negative mixed block graph having maximum one cut-vertex is called negative mixed star block graph. Let $\overline{m}K_n \setminus v_i$ denotes an induced subgraph resulting after vertex v_i is removed from $\overline{m}K_n$.

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Lemma 5.6. *The determinant of* $\overline{m}K_n \setminus v_i(i = 1, 2, \ldots, n)$ *is given by*

$$\Bigg(1 + \frac{1}{g_{n-1}}\Big(\sum_{i\le j} 2^{j-i} g_{i-1}h_{j+1} + \sum_{j<i} g_{j-1}h_{i+1}\Big)\Bigg)g_{n-1},$$

where

$$g_i = r_1 s_1^i + r_2 s_2^i, \quad \textit{for} \ \ i = 2, 3, \ldots, n-1,$$

$$h_i = r_{h1} s_1^{n-1-i} + r_{h2} s_2^{n-1-i}, \quad \textit{for} \ \ i = n-2, \ldots, 1,$$

$$r_1 = \frac{1}{2} + \frac{\iota}{2\sqrt{7}}, \quad r_2 = \frac{1}{2} - \frac{\iota}{2\sqrt{7}}, \quad r_{h1} = \frac{-1}{2} + \frac{3\iota}{2\sqrt{(7)}}, \quad r_{h2} = \frac{-1}{2} - \frac{3\iota}{2\sqrt{(7)}}, \ \textit{and}$$

$$s_1 = \frac{-1}{2} + \frac{\iota\sqrt{7}}{2}, \quad s_2 = \frac{-1}{2} - \frac{\iota\sqrt{7}}{2}.$$

Proof. Without loss of generality let us remove the first vertex v_1 of $\overline{m}K_n$. The adjacency matrix of $\overline{m}K_n \setminus v_1$ can be written as

$$A\Big(\overline{m}K_n \setminus v_1\Big) = \begin{bmatrix} 0 & -1 & 1 & \cdots & 1 \\ 0 & 0 & -1 & \ddots & 1 \\ 1 & 0 & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 1 & 1 & 1 & \ldots & 0 \end{bmatrix}.$$

Let $m = n - 1$. We can write, $A\Big(\overline{m}K_n \setminus v_1\Big) = uu^T + T$, where u is a $m \times 1$ column vector having all entries equal to 1. Here, T is the tridiagonal matrix of order m, having diagonal, subdiagonal entries equal to -1 and superdiagonal entries equal to -2.

The proof of this lemma can be found in [9] and an another expression of the proof can also be found in [8]. Using the matrix determinant lemma [5]

$$\det(T + uu^T) = (1 + u^T T^{-1} u)\det(T).$$

We need to solve some recursive expressions, in order to calculate the determinant and inverse of T [12]. We solve these recursive expressions using the roots of their characteristic equations. For the determinant of A, the recursive expression is

$$f_m = -f_{m-1} - 2f_{m-2}, \qquad f_0 = 1, \ f_{-1} = 0.$$

The roots of resulting characteristic equation $x^2 + x + 2 = 0$ are

$$s_1 = \frac{-1}{2} + \frac{\iota\sqrt{7}}{2}, \quad s_2 = \frac{-1}{2} - \frac{\iota\sqrt{7}}{2}.$$

Hence

$$\det(T) = f_m = r_1 s_1^m + r_2 s_2^m,$$

where using the initial conditions

$$r_1 = \frac{1}{2} + \frac{\iota}{2\sqrt{7}}, \quad r_2 = \frac{1}{2} - \frac{\iota}{2\sqrt{7}}.$$

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To calculate T^{-1} we need to solve the following recursive expressions

$$g_i = -g_{i-1} - 2g_{i-1}, \text{ for } i = 2, 3 \cdots, m, \ \ g_0 = 1, \ g_1 = -1$$

$$h_i = -h_{i+1} - 2h_{i+2}, \text{ for } i = m-1, \ldots, 1, \ \ h_{m+1} = 1, \ h_m = -1.$$

Similar to f_n, solving these recursive expressions we get

$$g_i = r_1 s_1^i + r_2 s_2^i, \text{ for } i = 2, 3, \ldots, n,$$

and,

$$h_i = r_{h1} s_1^{m-i} + r_{h2} s_2^{m-i}, \text{ for } i = m-1, \ldots, 1,$$

where

$$r_{h1} = \frac{-1}{2} + \frac{3\iota}{2\sqrt(7)}, \quad r_{h2} = \frac{-1}{2} - \frac{3\iota}{2\sqrt(7)}.$$

Entries of T^{-1} are clearly given by g_i, h_i [5].

$$T_{ij}^{-1} = \begin{cases} \frac{2^{j-i} g_{i-1} h_{j+1}}{g_m} & \text{if } i \leq j \\ \frac{g_{j-1} h_{i+1}}{g_m} & \text{if } j < i \end{cases}.$$

As, $u^T T^{-1} u$ equals to sum of all the entries of T^{-1}. Thus,

$$u^T T^{-1} u = \frac{1}{g_m} \Big(\sum_{i \leq j} 2^{j-i} g_{i-1} h_{j+1} + \sum_{j<i} g_{j-1} h_{i+1} \Big). \tag{5.6}$$

Hence, the determinant of $\overline{m}K_n \setminus v_i (i = 1, 2, \ldots, n)$ is given by

$$\Bigg(1 + \frac{1}{g_m} \Big(\sum_{i \leq j} 2^{j-i} g_{i-1} h_{j+1} + \sum_{j<i} g_{j-1} h_{i+1} \Big) \Bigg) g_{n-1}.$$

□

Theorem 5.7. *Let $\overline{m}G$ be mixed star negative block graph having k blocks $B_1, B_2, \ldots, B_k$ of order $n_1, n_2, \ldots, n_k$, respectively, then*

$$\det(\overline{m}G) = \sum_{i=1}^{k} \det(\overline{m}K_{n_i}) \prod_{j=1, j \neq i}^{k} D_n.$$

Proof. Proceeding as the proof of Theorem 5.4 the result directly follows by Lemma 5.3 and Lemma 3.2. □

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