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# SUFFICIENT CONDITIONS FOR TRIANGLE-FREE GRAPHS TO BE SUPER-$\lambda'$

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Abstract. An edge-cut $F$ of a connected graph $G$ is called a *restricted edge-cut* if $G-F$ contains no isolated vertices. The minimum cardinality of all restricted edge-cuts is called the *restricted edge-connectivity* $\lambda'(G)$ of $G$. A graph $G$ is said to be $\lambda'$-optimal if $\lambda'(G)=\xi(G)$, where $\xi(G)$ is the minimum edge-degree of $G$. A graph is said to be *super*-$\lambda'$ if every minimum restricted edge-cut isolates an edge.

In this paper, first, we provide a short proof of a previous theorem about the sufficient condition for $\lambda'$-optimality in triangle-free graphs, which was given in [J. Yuan and A. Liu, Sufficient conditions for $\lambda_k$-optimality in triangle-free graphs, *Discrete Math.*, **310** (2010) 981–987]. Second, we generalize a known result about the sufficient condition for triangle-free graphs being super-$\lambda'$ which was given by Shang et al. in [L. Shang and H. P. Zhang, Sufficient conditions for graphs to be $\lambda'$-optimal and super-$\lambda'$, *Networks*, **309** (2009) 3336–3345].

## 1. **Introduction**

We follow [7] for graph theoretic terminology and notation. Throughout the paper graphs are undirected finite connected without loops or multiple edges unless explicitly stated otherwise.

To measure the resilience of large networks, Esfahanian and Hakimi [2] introduced the vulnerability concept of restricted edge-connectivity. A set $F$ of edges of a connected graph $G$ is said to be a *restricted edge-cut*, if its removal disconnects $G$, and $G-F$ contains no isolated vertices. If $G$ has at

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least one restricted edge-cut, the *restricted edge-connectivity* of $G$, denoted by $\lambda'(G)$, is then defined to be the minimum cardinality over all restricted edge-cuts of $G$. The restricted edge-connectivity provides a more accurate measure of fault-tolerance of networks than the classical edge-connectivity (see [1]).

Let $G$ be a connected graph. For $e = xy \in E(G)$, let $\xi_G(e) = d_G(x) + d_G(y) - 2$ and $\xi(G) = \min\{\xi_G(e) \mid e \in E(G)\}$. The parameter $\xi(G)$ is called the *minimum edge-degree* of $G$. It has been shown in [2] that if a connected graph $G$ of order $n \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ is well-defined and $\lambda'(G) \leq \xi(G)$. For the convenience of statement, we call a graph which is not a disconnected graph, a triangle, or a star *$\lambda'$-connected graph*. A $\lambda'$-connected graph $G$ is called *$\lambda'$-optimal* if $\lambda'(G) = \xi(G)$. A $\lambda'$-connected graph $G$ is called *super restricted edge-connected*, in short, *super-$\lambda'$*, if any minimum restricted edge-cut isolates an edge.

For any pair of vertices $u, v$ of $G$, use $\mathrm{dist}(u, v)$ to denote the distance between $u$ and $v$. If $\mathrm{dist}(u, v) = d$, we also say that $u, v$ are two *distance-$d$* vertices. Moreover, define $\tau(G) = \min\{d_G(u)+d_G(v) \mid u, v \in V(G), \mathrm{dist}(u, v) = 2\}$.

In recent years, several authors studied the $\lambda'$-optimality of triangle-free graphs, and obtained some sufficient conditions for this class of graphs being $\lambda'$-optimal. Most of the conditions are related to the degree sum of any pair distance-2 vertices of the graph. For example, for a $\lambda'$-connected triangle-free graph $G$ of order $n$, Hellwig and Volkmannbit [3, 4] gave two sufficient conditions for $G$ being $\lambda'$-optimal: (1) If any pair of nonadjacent vertices of $G$ has at least two common neighbors, then it is $\lambda'$-optimal; (2) If $G$ is a bipartite graph such that $n \geq 10$, $\delta(G) \geq 3$ and $\tau(G) \geq 2\lfloor \frac{n+2}{4} \rfloor + 2$, then it is $\lambda'$-optimal. In [8], Yuan and Liu improved the second condition by showing that a $\lambda'$-connected triangle-free graph $G$ of order $n \geq 4$ is $\lambda'$-optimal if $\tau(G) \geq 2\lfloor \frac{n+2}{4} \rfloor + 1$. In [5], Shang and Zhang further showed that a $\lambda'$-connected bipartite graph $G$ of order $n \geq 4$ is super-$\lambda'$ if $\tau(G) \geq 2\lfloor \frac{n+2}{4} \rfloor + 3$.

In this paper, by analyzing the structure of $\lambda'$-fragments in a triangle-free graph $G$ of order $n$ with $\tau(G) \geq 2\lfloor \frac{n+2}{4} \rfloor + 1$, we first present a short proof of Yuan and Liu's result. Second, we improve Shang and Zhang's result by showing the following result.

**Theorem 1.1.** *For a connected triangle-free graph $G$ of order $n$, if $\tau(G) \geq 2\lfloor \frac{n+2}{4} \rfloor + 3$, then $G$ is super-$\lambda'$.*

## 2. Preliminary

Throughout this paper, denote by $K_{m,n}$ a complete bipartite graph, and by $K_{m,n} - e$ the graph obtained from $K_{m,n}$ by deleting an edge. Let $G$ be a connected graph. Use $|G|$ to represent the number of vertices of $G$. For $u, v \in V(G)$, $u \sim v$ means that $u$ and $v$ are adjacent and denote by $uv$ the edge incident to $u$ and $v$ in $G$. For any $v \in V(G)$ and $X \subseteq V(G)$, $N_X(v)$ is the set of vertices in $X$ adjacent to $v$, and if $X = V(G)$ then we also write $N(v)$ for $N_{V(G)}(v)$. For any subgraph $H$ of $G$

and $x \in V(G)$, use $d_H(x)$ to denote the number of vertices in $H$ adjacent to $x$, and if $H = G$ we also write $d(x)$ for $d_G(x)$.

Let $A$ be a subset of $V(G)$. The subgraph of $G$ induced by $A$ is denoted by $G[A]$. Setting $\overline{A} = V(G) \setminus A$, we rewrite $G[\overline{A}]$ as $G - A$. For the sake of convenience, we write $x$ for the single vertex set $\{x\}$. For two disjoint non-empty subsets $X$ and $Y$ of $V(G)$, let $[X, Y] = \{xy \in E(G) \mid x \in X, y \in Y\}$. If $Y = \overline{X}$, then we write $\partial(X)$ for $[X, \overline{X}]$ and $d(X) = |\partial(X)|$.

Let $G$ be a $\lambda'$-connected graph. A restricted edge-cut $F$ of $G$ is called a *$\lambda'$-edge cut* if $|F| = \lambda'(G)$. Let $A$ be a proper subset of $V(G)$. If $\partial(A)$ is a $\lambda'$-edge cut of $G$, then $A$ is called a *$\lambda'$-fragment* of $G$. It is clear that if $A$ is a $\lambda'$-fragment of $G$, then so is $\overline{A}$. Let

$$r(G) = \min\{|A| \mid A \text{ is a } \lambda' - \text{fragment of } G\}.$$

Obviously, $2 \le r(G) \le \frac{1}{2}|V(G)|$. A $\lambda'$-fragment $A$ is called a *$\lambda'$-atom* of $G$ if $|A| = r(G)$.

The following proposition gives a characterization of $\lambda'$-optimal graphs.

**Proposition 2.1.** [9, Theorem 1] *A $\lambda'$-connected graph $G$ is $\lambda'$-optimal if and only if $r(G) = 2$.*

For an optimal graph $G$, excluding all $\lambda'$-atoms, the smallest $\lambda'$-fragments with cardinality not larger than $|V(G)|/2$ in $G$, if any, are called *$\lambda'$-superatoms*. The following lemma is obvious (see also [6, Lemma 4]).

**Proposition 2.2.** *Let $G$ be a $\lambda'$-optimal graph. Then $G$ is super-$\lambda'$ if and only if it has no $\lambda'$-superatoms.*

Below, we give a technical lemma.

**Lemma 2.3.** *Let $G$ be a $\lambda'$-connected graph, and let $U$ be a $\lambda'$-fragment of $G$. Suppose that $G[U]$ has an edge, say $e = uv$, such that $|[\{u, v\}, U \setminus \{u, v\}]| \le |U \setminus \{u, v\}, \overline{U}|$. Then we have the following:*

(1) $\lambda'(G) = \xi(G) = \xi_G(e)$;

(2) $|[\{u, v\}, U \setminus \{u, v\}]| = |U \setminus \{u, v\}, \overline{U}|$.

*Proof.* Since $G$ is $\lambda'$-connected, one has $\lambda'(G) \le \xi(G) \le \xi_G(e)$. On the other hand,

$$\begin{aligned}
\xi(G) &\le \xi_G(e) \\
&= |[\{u, v\}, U \setminus \{u, v\}]| + |[\{u, v\}, \overline{U}]| \\
&\le |U \setminus \{u, v\}, \overline{U}| + |[\{u, v\}, \overline{U}]| \\
&= |[U, \overline{U}]| \\
&= \lambda'(G).
\end{aligned}$$

It follows that $\lambda'(G) = \xi(G) = \xi_G(e)$, and hence $|[\{u, v\}, U \setminus \{u, v\}]| = |U \setminus \{u, v\}, \overline{U}|$. □

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## 3. Proof of Theorem 1.1

For the convenience of statement, we call a connected triangle-free graph $G$ *Yuan-Liu graph* if $\tau(G) \geq 2\lfloor\frac{|G|+2}{4}\rfloor + 1$ (because Yuan & Liu first studied this classes of graphs in [8]). For a given Yuan-Liu graph $G$ of order $n$, we first give some properties of $\lambda'$-fragments $X$ of $G$ such that $3 \leq |X| \leq 2\lfloor\frac{n+2}{4}\rfloor + 1$. For notational convenience, throughout this section, we always use the following notations.

**Notations.**

- $G$: a nonsuper-$\lambda'$ Yuan-Liu graph of order $n$.
- $X$: a $\lambda'$-fragment of $G$ such that $|X| \geq 3$.
- $e = uv$: an edge in $G[X]$ such that $\xi_G(e) = \min\{\xi_G(f) \mid f \in E(G[X])\}$.
- $N_X^1(u) := \{x \mid x \in (N(u) \cap X) \setminus \{v\}, |[x, \overline{X}]| \geq 1\}$.
- $N_X^2(u) := \{x \mid x \in (N(u) \cap X) \setminus \{v\}, |[x, \overline{X}]| = 0\}$.
- $N_X^1(v) := \{y \mid y \in (N(v) \cap X) \setminus \{u\}, |[y, \overline{X}]| \geq 1\}$.
- $N_X^2(v) := \{y \mid y \in (N(v) \cap X) \setminus \{u\}, |[y, \overline{X}]| = 0\}$.
- $m = \lfloor\frac{n+2}{4}\rfloor$.

Since $G$ is triangle-free, we have the following two easy observations.

**Observation 1** *$N_X^1(u), N_X^2(u), N_X^1(v), N_X^2(v)$ are pairwise disjoint.*

**Observation 2** *For any two adjacent vertices $x, y$ of $G$, we have $N(x) \cap N(y) = \emptyset$.*

For each $x \in N_X(u) - v$, we have $d(x) \geq d(v)$, and so $d(x) \geq \lceil\frac{d(x)+d(v)}{2}\rceil \geq m + 1$. Similarly, $d(y) \geq m + 1$ for each $y \in N_X(v) - u$. So, we have the following.

**Observation 3** *For each $z \in N_X^1(u) \cup N_X^1(v) \cup N_X^2(u) \cup N_X^2(v)$, we have $d(z) \geq m + 1$.*

Below, we give several lemmas which reveal some properties of $X$.

**Lemma 3.1.** *Suppose that there are three vertices, say $x, y, z$, in $X$ such that $x \in \{u, v\}$ and $y \sim x$ and $z \sim y$. If $d(y) + d(z) \leq |X| \leq 2m + 1$, then $|X| = d(y) + d(z) = 2m + 1$ and $d(x) = d(y)$.*

*Proof.* Without loss of generality, assume that $x = u$. Since $G$ is triangle-free, one has $z \neq v$. This implies that $\text{dist}(v, y) = 2$ and $\text{dist}(x, z) = 2$ (see Fig. 1). Recall that $e = uv$ has minimum degree

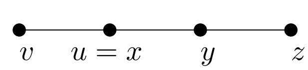


FIGURE 1. A path

over all edges in $G[X]$. So, $d(y) + d(z) \geq d(u) + d(v)$. If $d(y) < d(u)$, then $d(z) > d(v)$, and

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then $d(y) + d(z) > d(y) + d(v)$. However, $d(y) + d(v) \geq 2m + 1$ since $\text{dist}(y, v) = 2$. This means that $|X| > 2m + 1$, a contradiction. Thus, $d(y) \geq d(u)$, and hence $d(y) + d(z) \geq d(u) + d(z)$. Since $\text{dist}(z, u) = 2$, one has $d(u)+d(z) \geq 2m+1$. This means that $|X| \geq d(y)+d(z) \geq d(u)+d(z) \geq 2m+1$. As it is assumed that $|X| \leq 2m+1$, one has $|X| = d(y)+d(z) = 2m+1$, forcing $d(y) = d(u)$, namely, $d(y) = d(x)$. The proof is completed. □

**Lemma 3.2.** *If $N_X^2(u) \neq \emptyset$ or $N_X^2(v) \neq \emptyset$, then $|X| \geq 2m + 1$.*

*Proof.* Without loss of generality, assume that $N_X^2(u) \neq \emptyset$, and take $w \in N_X^2(u)$. Suppose to the contrary that $|X| \leq 2m$. By Observation 2, $N(w) \cap (N_X^1(u) \cup N_X^2(u)) = \emptyset$.

Suppose $N(w)\cap N_X^2(v) \neq \emptyset$. Take $w' \in N(w)\cap N_X^2(v)$. By Observation 2, we get $N(w)\cap N(w') = \emptyset$ and by Observation 3, we obtain $|N(w) \cup N(w')| \geq d(w) + d(w') \geq 2m + 2$. Since $w \in N_X^2(u)$ and $w' \in N_X^2(v)$, one has $N(w) \cup N(w') \subseteq X$, implying $|X| \geq 2m + 2$, a contradiction.

Now suppose $N(w) \cap N_X^2(v) = \emptyset$. Since $X$ is a $\lambda'$-fragment, one has $d(X) \leq |\partial(\{u, v\})|$. It follows that

$$|[\{u, v\}, \overline{X}]| + |[X \setminus \{u, v\}, \overline{X}]| \leq |[\{u, v\}, \overline{X}]| + |[\{u, v\}, X \setminus \{u, v\}]|,$$

and hence $|[X \setminus \{u, v\}, \overline{X}]| \leq |[\{u, v\}, X \setminus \{u, v\}]|$. This gives that

$$|[X \setminus \{u, v\}, \overline{X}]| \leq |N_X^1(u)| + |N_X^2(u)| + |N_X^1(v)| + |N_X^2(v)|. \tag{3.1}$$

If $|w', \overline{X}| \geq 2$ for each $w' \in N(w) - u$, then

$$2|N(w) \setminus \{u\}| + |N_X^1(u)| + |N_X^1(v) \setminus N(w)| \leq |[X \setminus \{u, v\}, \overline{X}]|.$$

Combining this with Eq. (3.1) and Observation 3, we obtain

$$2m \leq |N(w) \setminus \{u\}| \leq |N_X^2(u)| + |N_X^2(v)| + |N_X^1(v) \cap N(w)|. \tag{3.2}$$

Set $Y = N_X^2(u) \cup N_X^2(v) \cup (N_X^1(v) \cap N(w)) \cup \{v\}$. Clearly, $Y \subseteq X$ and $|Y| = |N_X^2(u)| + |N_X^2(v)| + |N_X^1(v) \cap N(w)| + |\{v\}|$. As a result, $|X| \geq |Y| \geq 2m + 1$, a contradiction. Thus, there must exist $w' \in N(w)-u$ such that $|[w', \overline{X}]| \leq 1$. If $d(w)+d(w') \leq |X|$ then by Lemma 3.1, $|X| = d(w)+d(w') = 2m + 1$, a contradiction. This implies that $|[w', \overline{X}]| = 1$ and $X = N(w) \cup N_X(w')$. Since $u \in N(w)$, one has $v \in N(w')$. So, $w' \in N_X^1(v)$ and hence $d(w') \geq m + 1$ by Observation 3. Consequently, $|X| \geq d(w) + d(w') - 1 \geq 2m + 1$, a contradiction. □

**Lemma 3.3.** *If $N_X^2(u) = N_X^2(v) = \emptyset$, then $\lambda'(G) = \xi_G(e) = \xi(G)$, and*

$$|N_X^1(u)| + |N_X^1(v)| = |[X \setminus \{u, v\}, \overline{X}]|, \textit{and } |[z, \overline{X}]| = 1, \forall z \in N_X^1(u) \cup N_X^1(v).$$

*Proof.* Since $N_X^2(u) = N_X^2(v) = \emptyset$, one has $|[\{u, v\}, X \setminus \{u, v\}]| = |N_X^1(u)| + |N_X^1(v)|$. Clearly,

$$|N_X^1(u)| + |N_X^1(v)| \leq |[N_X^1(u), \overline{X}]| + |[N_X^1(v), \overline{X}]| \leq |[X \setminus \{u, v\}, \overline{X}]|.$$

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It follows that $|[\{u,v\}, X\setminus\{u,v\}]| \le |[X\setminus\{u,v\}, \overline{X}]|$. By Lemma 2.3, we get that $\lambda'(G) = \xi(G) = \xi_G(e)$, and $|[\{u,v\}, X\setminus\{u,v\}]| = |[X\setminus\{u,v\}, \overline{X}]|$. In particular, $|N_X^1(u)| + |N_X^1(v)| = |[X\setminus\{u,v\}, \overline{X}]|$. Observe that each vertex in $N_X^1(u) \cup N_X^1(v)$ has at least one neighbor in $\overline{X}$. We have $|[z, \overline{X}]| = 1, \forall z \in N_X^1(u) \cup N_X^1(v)$. □

By Lemmas 3.2 and 3.3, we can obtain the following corollary which was first given in [8].

**Corollary 3.4.** [8, Theorem 2.3] *Let $G$ be a Yuan-Liu graph. Then $G$ is $\lambda'$-optimal.*

*Proof.* Let $X$ be a $\lambda'$-fragment of $G$ such that $3 \le |X| \le n/2$. Since $2m+1 > n/2$, by Lemma 3.2, we have $N_X^2(u) = N_X^2(v) = \emptyset$. Applying Lemma 3.3, $G$ is $\lambda'$-optimal. □

**Lemma 3.5.** *If $|X| \le 2m$, then $G[X] \cong K_{m,m}$.*

*Proof.* From Lemma 3.2 we obtain that $N_X^2(u) = N_X^2(v) = \emptyset$, and by Lemma 3.3, $[N_X^1(u), \overline{X}] \cup [N_X^1(v), \overline{X}] = [X\setminus\{u,v\}, \overline{X}]$.

Suppose that there exists $w \in N_X^1(u) \cup N_X^1(v)$ such that $N_X(w) \subsetneq N_X(u) \cup N_X(v)$. Without loss of generality, assume that $w \in N_X^1(u)$. Take $w' \in N_X(w) \setminus N_X(u) \cup N_X(v)$. Then, $|[w', \overline{X}]| = 0$, and so $N(w') \subseteq X$. This implies that $\{u,v\} \cap N(w') = \emptyset$. By Observation 2, we have $N(w') \cap N(w) = \emptyset$. Clearly, $v \notin N_X(w)$. It follows that $|X| \ge |N_X(w)| + |\{v\}| + |N(w')|$. By Lemma 3.3, we have $d(w) = |N_X(w)| + 1$, and hence $|X| \ge d(w) + d(w')$. From Lemma 3.1 it follows that $|X| = d(w) + d(w') = 2m+1$, a contradiction. Hence, for any $w \in N_X^1(u) \cup N_X^1(v)$, we have $N_X(w) \subseteq N_X(u) \cup N_X(v)$. Consequently, $X = N_X(u) \cup N_X(v)$ because $G[X]$ is connected. Since $G$ is triangle-free, $G[X]$ is a bipartite graph. Without loss of generality, assume that $|N_X(u)| \le |N_X(v)|$. Since $|X| \le 2m$, one has $|N_X(u)| \le m$. Take $x \in N_X(v)$. Then $N_X(x) \subseteq N_X(u)$. By Observation 3, $d(x) \ge m+1$ and by Lemma 3.3, $d_X(x) \ge m$. So, $m \le d_X(x) = |N_X(x)| \le |N_X(u)| = d_X(u) \le m$. It follows that $N_X(x) = N_X(u)$ which has cardinality $m$. So, $|N_X(v)| = m$. By the arbitrariness of $x$, we get that $G[X] \cong K_{m,m}$. □

Now, we are ready to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1** Suppose that $G$ is nonsuper-$\lambda'$. By Proposition 2.2, $G$ has a $\lambda'$-fragment, say $X$, such that $3 \le |X| \le n/2$. From Lemma 3.5 it follows that $G[X] \cong K_{m,m}$, where $m = \lfloor \frac{n+2}{4} \rfloor$. Choose an edge, say $e = uv$, in $E(G[X])$ such that $\xi_G(e)$ =min$\{\xi_G(f) \mid f \in E(G[X])\}$. By Lemma 3.3, for each $w \in X\setminus\{u,v\}$, $|[w, \overline{X}]| = 1$, and $\xi_G(e) = d(X) \ge 2m-2$. Since $|X| \ge 3$, one has $m > 1$. So, we can take $e' = xy \in E(G(X))$ having no common vertices with $e$. Clearly, $d(x) = d(y) = m+1$, so $\xi_G(e') = 2m$. Hence, $2m-2 \le \xi_G(e) \le 2m$. This implies that at least one of $u, v$ has degree at most $m+1$. Without loss of generality, assume $d(u) \le m+1$. As $G[X] \cong K_{m,m}$, we may assume that $\mathrm{dist}(x,u) = 2$. Then $d(u) + d(x) \le 2m+2$, contrary the assumption that $\tau(G) \ge 2m+3$. □

## 4. **Remarks**

To show Theorem 1.1 is optimal, in this section we shall construct two families of nonsuper-$\lambda'$ Yuan-Liu graphs. For an integer $m > 1$, let $H_1, H_2$ be two disjoint copies of $K_{m,m}$.

**Definition 4.1.** *The graph $\mathcal{G}_\ell^{1,2}$ is a graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2) \cup I_\ell^{1,2}$, where $I_\ell^{1,2}$ is a set of $\ell$ independent edges with one endpoint in $V(H_1)$ and the other endpoint in $V(H_2)$, where $\ell = 2m$ or $2m-1$.*

**Lemma 4.2.** *For each $\ell = 2m$ or $2m-1$, the graph $\mathcal{G}_\ell^{1,2}$ is a nonsuper-$\lambda'$ Yuan-Liu graph. Furthermore, $\tau(\mathcal{G}_{2m}^{1,2}) = 2m+2$ and $\tau(\mathcal{G}_{2m-1}^{1,2}) = 2m+1$.*

*Proof.* Since $I_\ell^{1,2}$ is a matching, $\mathcal{G}_\ell^{1,2}$ is triangle-free. Clearly, $|\mathcal{G}_\ell^{1,2}| = 4m$, so $2\lfloor\frac{|\mathcal{G}_\ell^{1,2}|+2}{4}\rfloor + 1 = 2m+1$. If $\ell = 2m$, then $\mathcal{G}_{2m}^{1,2}$ is a $(m+1)$-regular graph, and so $\tau(\mathcal{G}_{2m}^{1,2}) = 2m+2$ and $\xi(\mathcal{G}_{2m}^{1,2}) = 2m$. If $\ell = 2m-1$, then $\mathcal{G}_{2m-1}^{1,2}$ has exactly two vertices, say $u_1, u_2$, of degree $m$ such that $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$, and the remaining vertices have degree $m+1$. Clearly, $\mathrm{dist}(u_1, u_2) \geq 5$. This implies that $\tau(\mathcal{G}_{2m-1}^{1,2}) = 2m+1$ and $\xi(\mathcal{G}_{2m-1}^{1,2}) = 2m-1$. Thus, $\mathcal{G}_\ell^{1,2}$ is a Yuan-Liu graph for each $\ell = 2m$ or $2m-1$. By Corollary 3.4, $\mathcal{G}_\ell^{1,2}$ is $\lambda'$-optimal. Clearly, $I_\ell^{1,2}$ is a $\lambda'$-edge cut. So, $\mathcal{G}_\ell^{1,2}$ is nonsuper-$\lambda'$. □

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