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THE ANNIHILATOR GRAPH OF A 0-DISTRIBUTIVE LATTICE

SAEID BAGHERI* AND MAHTAB KOOHI KERAHROODI

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Abstract. In this article, for a lattice $\mathcal{L}$, we define and investigate the annihilator graph $\mathfrak{ag}(\mathcal{L})$ of $\mathcal{L}$ which contains the zero-divisor graph of $\mathcal{L}$ as a subgraph. Also, for a 0-distributive lattice $\mathcal{L}$, we study some properties of this graph such as regularity, connectedness, the diameter, the girth and its domination number. Moreover, for a distributive lattice $\mathcal{L}$ with $Z(\mathcal{L}) \neq \{0\}$, we show that $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$ if and only if $\mathcal{L}$ has exactly two minimal prime ideals. Among other things, we consider the annihilator graph $\mathfrak{ag}(\mathcal{L})$ of the lattice $\mathcal{L} = (\mathcal{D}(n), |)$ containing all positive divisors of a non-prime natural number n and we compute some invariants such as the domination number, the clique number and the chromatic number of this graph. Also, for this lattice we investigate some special cases in which $\mathfrak{ag}(\mathcal{D}(n))$ or $\Gamma(\mathcal{D}(n))$ are planar, Eulerian or Hamiltonian.

1. Introduction

For a commutative ring R with nonzero identity, let $Z(R)$ be the set of zero-divisors. D.F. Anderson and P. Livingston [5] introduced the zero-divisor graph of R, denoted by $\Gamma(R)$, as the (undirected) graph with vertex set $Z(R)^\star := Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. (The original definition has been appeared in Beck [9] and Anderson and Naseer [4] considering all elements of the ring R as the vertex set). This graph has been extensively studied and investigated by several authors (see for example [2], [4], [5], [7], [9] and [22]). For a survey article about the zero-divisor graphs, the reader is refered to [11].

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∗Corresponding author.

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This idea has been used and generalized in many different directions and many authors defined different graphs associated to algebraic structures in order to investigate the interplay between algebraic features of an algebraic structure and graph theoretic properties of the corresponding graph. The zero-divisor graph of a semigroup with a zero element has been defined by F. R. Demeyer et al. [12]. The total graph of a commutative ring, defined by D. F. Anderson and A. Badawi [6], is another graph associated to a commutative ring. A new extension of the zero-divisor graph is the concept annihilator graph $AG(R)$ of a commutative ring R, defined by A. Badawi [8] which contains the zero-divisor graph of R as a subgraph. The vertex set of this graph is $Z(R)^\star := Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $\text{ann}_R(xy) \neq \text{ann}_R(x) \cup \text{ann}_R(y)$. S. Dutta and Ch. Lanong [14] investigated the annihilator graph of a finite commutative ring. They determined some conditions on a ring under which its annihilator graph is complete or regular. Moreover, for most recent study in this direction see [1] and [23].

There are many papers which interlink graph theory and lattice theory(see for example [10, 13, 15, 16, 17]). These papers discuss the properties of graphs derived from partially ordered sets and lattices. The zero-divisor graph has been defined and investigated for lattices by some authors such as E. Estaji and K. Khashyarmanesh [15] and V. Joshi and S. Sarode [21] and others. In fact, for a lattice $\mathcal{L}$ with the bottom element 0, let $Z(\mathcal{L}) := \{x \in \mathcal{L} | \exists a \in \mathcal{L} \setminus \{0\}; \quad x \wedge a = 0\}$. Then the *zero divisor graph* of $\mathcal{L}$ is the (undirected) graph with vertex set $Z(\mathcal{L})^\star := Z(\mathcal{L}) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $x \wedge y = 0$.

Let $\mathcal{L}$ be a lattice with the bottom element 0 and for $a \in Z(\mathcal{L})$, let $\text{ann}_{\mathcal{L}}(a) = \{r \in \mathcal{L} | \, r \wedge a = 0\}$. In this article, we introduce and investigate the *annihilator graph* $\mathfrak{ag}(\mathcal{L})$ of $\mathcal{L}$ as the (undirected) graph with vertex set $Z(\mathcal{L})^\star = Z(\mathcal{L}) \setminus \{0\}$ and two distinct vertices x and y are considered to be adjacent if and only if $\text{ann}_{\mathcal{L}}(x \wedge y) \neq \text{ann}_{\mathcal{L}}(x) \cup \text{ann}_{\mathcal{L}}(y)$. In fact, this graph contains the zero-divisor graph $\Gamma(\mathcal{L})$ as a subgraph. We study some basic properties of $\mathfrak{ag}(\mathcal{L})$ such as the diameter, the girth, the domination number and the regularity of this graph. In section 2, it is shown that for a 0-distributive lattice $\mathcal{L}$, the annihilator graph $\mathfrak{ag}(\mathcal{L})$ is connected and its diameter is at most 2 (see Corollary 2.5). Among other things, we prove that for a 0-distributive lattice $\mathcal{L}$, if $\mathfrak{ag}(\mathcal{L})$ is not identical to $\Gamma(\mathcal{L})$, then $\text{gr}(\mathfrak{ag}(\mathcal{L})) = 3$ (see Lemma 2.8). Also, if a bounded lattice $\mathcal{L}$ of finite length can be decomposed as a product of at least two nonzero lattices, then we find a small upper bound for the domination number of $\mathfrak{ag}(\mathcal{L})$. Namely, we show that $\text{dt}(\mathfrak{ag}(\mathcal{L})) \leq 2$ (see Theorem 2.12). This gives a (generalized) lattice counterpart for [14, Proposition 2.6]. Furthermore, we show that for a nontrivial finite lattice $\mathcal{L}$, if $\mathfrak{ag}(\mathcal{L})$ is a regular graph and $\Pi_{j \neq i}|\mathcal{L}_j| \nmid |Z(\mathcal{L}_i)^\star|$, then $\mathcal{L}$ is a product of at most two directly indecomposable lattices, or $\mathcal{L} \cong C_2 \times C_2 \times C_2$, where C_2 is the two-element chain (see Theorem 2.11).

In section 3, we determine when $\mathfrak{ag}(\mathcal{L})$ and $\Gamma(\mathcal{L})$ are the same. In fact, we prove that for a distributive lattice $\mathcal{L}$ with $Z(\mathcal{L}) \neq \{0\}$ the annihilator graph $\mathfrak{ag}(\mathcal{L})$ is identical to the zero-divisor graph $\Gamma(\mathcal{L})$ if and only if $\mathcal{L}$ has precisely two minimal prime ideals. Also, we show when $\mathfrak{ag}(\mathcal{L})$ is a

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complete, a complete bipartite or a star graph. In section 4, we consider the annihilator graph $\mathfrak{ag}(\mathcal{L})$ of the lattice $\mathcal{L} = (\mathcal{D}(n), |)$ containing all positive divisors of a non-prime natural number n and we show that for this lattice if n has at least three distinct prime divisors, then $\mathfrak{ag}(\mathcal{L}) \neq \Gamma(\mathcal{L})$. Moreover, we compute some invariants such as the domination number, the clique number and the chromatic number of the graph $\mathfrak{ag}(\mathcal{L})$ associated to this lattice. Also, in this special case, we investigate some more special cases in which $\mathfrak{ag}(\mathcal{D}(n))$ or $\Gamma(\mathcal{D}(n))$ are planar, Eulerian or Hamiltonian (see Theorem 4.3).

We recall some preliminary definitions, notations and properties of graphs and lattices. For undefined concepts in graph and lattice theory the reader is refered to [26, 27] and [18] respectively. A graph without any vertices (resp. edges) is called an *empty graph* (a *null graph*). A *complete r-partite graph* is a graph whose vertex set is partitioned into r separated subsets such that each vertex is joined to every other vertex that is not in the same subset. For positive integers m and n, the graph K^n is a complete graph with n vertices and the graph $K^{m,n}$ is a complete bipartite graph, with parts of sizes m and n. A complete bipartite graph $K^{1,n}$ is called a *star graph*. A graph G is said to be *connected* if there is a path between every two distinct vertices. Also, G is called a regular graph if the degree of all vertices in G are the same. A nonempty subset S of $V(G)$ is called a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\mathrm{dt}(G)$ of G is the minimum cardinality of dominating sets in G. If $\mathrm{dt}(G) = \gamma$, then every dominating set with cardinality γ is called a γ-set of G. A *clique* of a graph G is a complete subgraph of G and the maximum size of cliques in G is called the *clique number* of G and is denoted by $\omega(G)$.

For two elements a and b in a lattice $(\mathcal{L}, \leq, \wedge, \vee)$, we say that b *covers* a and write $a \prec b$ if $a \lneq b$ and there is no element x in $\mathcal{L}$ such that $a \lneq x \lneq b$. Let $\mathcal{L}$ be a bounded lattice. An element $a \in \mathcal{L}$ is called an *atom* (resp. *coatom*) if $0 \prec a$ (resp. $a \prec 1$). The set of all atoms in $\mathcal{L}$ is denoted by $\mathcal{A}(\mathcal{L})$. A subset I of a lattice $\mathcal{L}$ is called an *ideal* if I contains the join of each pair of elements in I and for every $x \in I$ and $a \in \mathcal{L}$, $a \wedge x \in I$. A *filter* in $\mathcal{L}$ is defined dually. For every element $a \in \mathcal{L}$, the principal filter (ideal) generated by a is denoted by $[a)$ $((a])$. A proper ideal P of a lattice $\mathcal{L}$ is called a *prime ideal* if for each $a, b \in \mathcal{L}$, $a \wedge b \in P$ implies that $a \in P$ or $b \in P$.

2. Some fundamental Properties of $\mathfrak{ag}(\mathcal{L})$

Let $\mathcal{L}$ be a lattice with the bottom element 0. We recall that the annihilator graph $\mathfrak{ag}(\mathcal{L})$ of $\mathcal{L}$ is the (undirected) graph whose vertex set is $Z(\mathcal{L})^\star = Z(\mathcal{L}) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_{\mathcal{L}}(x \wedge y) \neq \mathrm{ann}_{\mathcal{L}}(x) \cup \mathrm{ann}_{\mathcal{L}}(y)$. In this section we state some basic results about the annihilator graph $\mathfrak{ag}(\mathcal{L})$ which are used in the next sections.

Definition 2.1. *A lattice $\mathcal{L}$ with the bottom element 0 is said to be* 0-distributive *if the equalities $a \wedge b = 0 = a \wedge c$ imply that $a \wedge (b \vee c) = 0$.*

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Example 2.2. *As a well-known lattice, we consider* $\mathcal{L} := \mathcal{D}(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ *containing all natural divisors of* 30 *which is a distributive lattice with the order defined by divisibility. For every* $x, y \in \mathcal{L}$*, the meet* $x \wedge y$ *and the join* $x \vee y$ *are defined to be the greatest common divisor and the least common multiple of* x *and* y *respectively. As we see in the following figures, the zero-divisor graph* $\Gamma(\mathcal{L})$ *is a* ***proper*** *subgraph of the annihilator graph* $\mathfrak{ag}(\mathcal{L})$*. For example,*

$$\{1, 3, 5, 15\} = \text{ann}_{\mathcal{L}}(2) = \text{ann}_{\mathcal{L}}(\gcd(6, 10)) \neq \text{ann}_{\mathcal{L}}(6) \cup \text{ann}_{\mathcal{L}}(10) = \{1, 3, 5\}.$$

Therefore, the vertices 6 *and* 10 *are adjacent in* $\mathfrak{ag}(\mathcal{L})$ *but not in* $\Gamma(\mathcal{L})$*. For an investigation of the graphs associated to this kind of lattices in a more general case, see Theorem 4.3.*

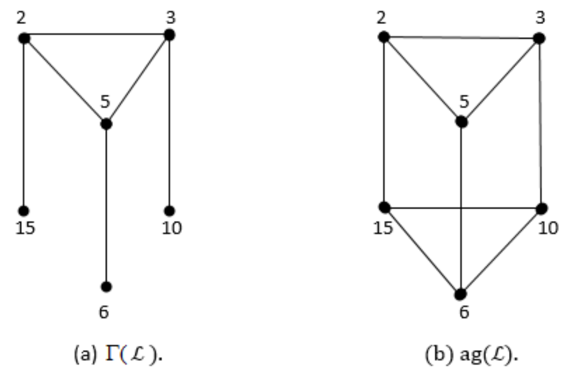

Fig. 1. $\Gamma(\mathcal{L})$ is not equal with $\text{ag}(\mathcal{L})$.

Lemma 2.3. *Let* $\mathcal{L}$ *be a 0-distributive lattice and* x *and* y *be distinct elements in* $Z(\mathcal{L})^\star$*. Then the following statements hold:*

(1) *If* $x \in \mathcal{L}$*, then* $\text{ann}_{\mathcal{L}}(x)$ *is an ideal of* $\mathcal{L}$*.*

(2) x *and* y *are not adjacent in* $\mathfrak{ag}(\mathcal{L})$ *if and only if* $\text{ann}_{\mathcal{L}}(x) \subseteq \text{ann}_{\mathcal{L}}(y)$ *or* $\text{ann}_{\mathcal{L}}(y) \subseteq \text{ann}_{\mathcal{L}}(x)$*.*

(3) x *and* y *are not adjacent in* $\mathfrak{ag}(\mathcal{L})$ *if and only if* $\text{ann}_{\mathcal{L}}(x \wedge y) = \text{ann}_{\mathcal{L}}(x)$ *or* $\text{ann}_{\mathcal{L}}(x \wedge y) = \text{ann}_{\mathcal{L}}(y)$*.*

(4) *If* $x - y$ *is an edge of* $\Gamma(\mathcal{L})$*, then* $x - y$ *is an edge of* $\mathfrak{ag}(\mathcal{L})$*. In fact,* $\Gamma(\mathcal{L})$ *is subgraph of* $\mathfrak{ag}(\mathcal{L})$*.*

(5) *If* $d_{\Gamma(\mathcal{L})}(x, y) = 3$*, then* $x - y$ *is an edge of* $\mathfrak{ag}(\mathcal{L})$*.*

(6) *If* x *and* y *are not adjacent in* $\mathfrak{ag}(\mathcal{L})$*, then there is a vertex* $\omega \in Z(\mathcal{L})^\star \setminus \{x, y\}$ *such that* $x - \omega - y$ *is a path in* $\Gamma(\mathcal{L})$ *and hence* $x - \omega - y$ *is also a path in* $\mathfrak{ag}(\mathcal{L})$*.*

Proof. (1) Let $l_1, l_2 \in \text{ann}_{\mathcal{L}}(x)$. Then $l_1 \wedge x = 0 = l_2 \wedge x$. Therefore, $(l_1 \vee l_2) \wedge x = 0$, i.e. $l_1 \vee l_2 \in \text{ann}_{\mathcal{L}}(x)$.

(2) Suppose that x is not adjacent to y in $\mathfrak{ag}(\mathcal{L})$. Then $\text{ann}_{\mathcal{L}}(x) \cup \text{ann}_{\mathcal{L}}(y) = \text{ann}_{\mathcal{L}}(x \wedge y)$. If $\text{ann}_{\mathcal{L}}(x) \nsubseteq \text{ann}_{\mathcal{L}}(y)$, then we take an element $a \in \mathcal{L}$ such that $a \wedge x = 0$ but $a \wedge y \neq 0$. Now for every $b \in \text{ann}_{\mathcal{L}}(y)$, $b \vee a \in \text{ann}_{\mathcal{L}}(x \wedge y) = \text{ann}_{\mathcal{L}}(x) \cup \text{ann}_{\mathcal{L}}(y)$. This implies that $b \vee a \in \text{ann}_{\mathcal{L}}(x)$

or $b \vee a \in \mathrm{ann}_{\mathcal{L}}(y)$. The second possibility yields a contradiction and the first one implies that $b \in \mathrm{ann}_{\mathcal{L}}(x)$. This means that $\mathrm{ann}_{\mathcal{L}}(y) \subseteq \mathrm{ann}_{\mathcal{L}}(x)$.

Conversely, assume that $\mathrm{ann}_{\mathcal{L}}(x) \subseteq \mathrm{ann}_{\mathcal{L}}(y)$. On contrary suppose that $\mathrm{ann}_{\mathcal{L}}(x) \cup \mathrm{ann}_{\mathcal{L}}(y) \neq \mathrm{ann}_{\mathcal{L}}(x \wedge y)$. Thus $\mathrm{ann}_{\mathcal{L}}(x \wedge y) \nsubseteq \mathrm{ann}_{\mathcal{L}}(y)$. Therefore, there exists an element $z \in \mathrm{ann}_{\mathcal{L}}(x \wedge y)$ such that $z \wedge y \neq 0$ and so $z \wedge y \in \mathrm{ann}_{\mathcal{L}}(x) \subseteq \mathrm{ann}_{\mathcal{L}}(y)$ which is a contradiction.

(3) By part (2) is trivial.

(4) Suppose that $x - y$ is an edge of $\Gamma(\mathcal{L})$. Then $x \wedge y = 0$ and hence $\mathrm{ann}_{\mathcal{L}}(x \wedge y) = \mathcal{L}$. Since $(x \vee y) \wedge x = x \neq 0$ and $(x \vee y) \wedge y = y \neq 0$, $\mathrm{ann}_{\mathcal{L}}(x \wedge y) \neq \mathrm{ann}_{\mathcal{L}}(x) \cup \mathrm{ann}_{\mathcal{L}}(y)$. Thus we must have $x - y$ as an edge in $\mathfrak{ag}(\mathcal{L})$.

(5) Suppose that $d_{\Gamma(\mathcal{L})}(x, y) = 3$. Then there are $\alpha, \beta \in Z(\mathcal{L})^\star$ such that $x \wedge \alpha = 0, y \wedge \beta = 0$ and $x \wedge \beta \neq 0, y \wedge \alpha \neq 0$. Therefore $\mathrm{ann}_{\mathcal{L}}(x) \nsubseteq \mathrm{ann}_{\mathcal{L}}(y)$ and $\mathrm{ann}_{\mathcal{L}}(y) \nsubseteq \mathrm{ann}_{\mathcal{L}}(x)$. Hence $x - y$ is an edge of $\mathfrak{ag}(\mathcal{L})$ by part (2).

(6) Suppose that x and y are not adjacent in $\mathfrak{ag}(\mathcal{L})$. Then there is a nonzero element $\omega \in \mathrm{ann}_{\mathcal{L}}(x) \cap \mathrm{ann}_{\mathcal{L}}(y)$, by (2). Since $x \wedge y \neq 0$, we have $\omega \in Z(\mathcal{L})^\star \setminus \{x, y\}$. Hence $x - \omega - y$ is a path in $\Gamma(\mathcal{L})$, and thus $x - \omega - y$ is a path in $\mathfrak{ag}(\mathcal{L})$ by (4).

□

We recall that for vertices x and y in a graph G, let $d(x, y)$ be the length of a shortest path from x to y. The *diameter* of G is $\mathrm{diam}(G) = \sup\{d(x, y) : x, y \in V(G)\}$. It is well-known that for a lattice $\mathcal{L}$ with 0, $\mathrm{diam}(\Gamma(\mathcal{L})) \leq 3$ (see [19]). The following corollaries can be easily obtained from parts (4),(5) and (6) of Lemma 2.3.

Corollary 2.4. *For a* 0*-distributive lattice* $\mathcal{L}$*, if* $d_{\mathfrak{ag}(\mathcal{L})}(x, y) = 2$*, then* $d_{\Gamma(\mathcal{L})}(x, y) = 2$*.*

Corollary 2.5. *Let* $\mathcal{L}$ *be a* 0*-distributive lattice with* $Z(\mathcal{L}) \neq \{0\}$*. Then* $\mathfrak{ag}(\mathcal{L})$ *is connected and its diameter is at most* 2*.*

It has been shown in [15, Proposition 3.8] that every isomorphism $f : \mathcal{L} \to \mathcal{S}$ between finite lattices induces a graph isomorphism $f : V(\Gamma(\mathcal{L})) \to V(\Gamma(\mathcal{S}))$ between the zero-divisor graphs associated to these lattices. As we state in the following proposition, a similar result is also true for the annihilator graphs.

Proposition 2.6. *Let* $\mathcal{L}$ *and* $\mathcal{S}$ *be lattices with bottom element 0. If* $f : \mathcal{L} \to \mathcal{S}$ *is a lattice isomorphism, then* $f|_{Z(\mathcal{L})^\star} : V(\mathfrak{ag}(\mathcal{L})) \to V(\mathfrak{ag}(\mathcal{S}))$ *defines a graph isomorphism.*

Proof. Let $a - b$ be an edge in $\mathfrak{ag}(\mathcal{L})$. This means that there exists an element $x \in \mathcal{L}$ such that $x \wedge a \wedge b = 0$ and $x \wedge a \neq 0$ and $x \wedge b \neq 0$. Thus, $f(x) \in \mathrm{ann}_{\mathcal{S}}(f(a) \wedge f(b)) \setminus (\mathrm{ann}_{\mathcal{S}}(f(a)) \cup \mathrm{ann}_{\mathcal{S}}(f(b)))$, as desired. □

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Remark 2.7. *As it has been shown in* [15, Example 3.9], *the converse of Proposition 2.6 does not hold. In fact, if we consider* $\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ *and* $\mathcal{S} = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$ *with two operations* $\cap$ *and* $\cup$, *then* $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L}) = \Gamma(\mathcal{S}) = \mathfrak{ag}(\mathcal{S})$ *but* $\mathcal{L} \ncong \mathcal{S}$.

In some important special cases, the following lemma and its corollary help us to determine when $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$. They will be also used to find the girth of the annihilator graph $\mathfrak{ag}(\mathcal{L})$ (see Theorems 3.9 and 3.11).

Lemma 2.8. *Let* $\mathcal{L}$ *be a lattice with bottom element* 0 *and there exist two distinct elements* $x, y \in Z(\mathcal{L})^\star$ *which are adjacent in* $\mathfrak{ag}(\mathcal{L})$ *but they are not adjacent in* $\Gamma(\mathcal{L})$. *Then* $\mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = 3$ *and there is a cycle* C *in* $\mathfrak{ag}(\mathcal{L})$ *in which no pair of distinct vertices are adjacent in* $\Gamma(\mathcal{L})$.

Proof. Suppose that $x - y$ is an edge of $\mathfrak{ag}(\mathcal{L})$ which is not an edge of $\Gamma(\mathcal{L})$. Then $x \wedge y \neq 0$ and there is an $\omega \in \mathrm{ann}_{\mathcal{L}}(x \wedge y) \setminus \{x, y\}$ such that $\omega \wedge x \neq 0$ and $\omega \wedge y \neq 0$. Now $y \in \mathrm{ann}_{\mathcal{L}}(x \wedge \omega) \setminus (\mathrm{ann}_{\mathcal{L}}(x) \cup \mathrm{ann}_{\mathcal{L}}(\omega))$, implies that $x - \omega$ is an edge of $\mathfrak{ag}(\mathcal{L})$. Similarly, $x \in \mathrm{ann}_{\mathcal{L}}(y \wedge \omega) \setminus (\mathrm{ann}_{\mathcal{L}}(y) \cup \mathrm{ann}_{\mathcal{L}}(\omega))$ yields that $y - \omega$ is an edge of $\mathfrak{ag}(\mathcal{L})$. Hence $x - \omega - y$ is a path in $\mathfrak{ag}(\mathcal{L})$ which is not a path in $\Gamma(\mathcal{L})$. It is clear that $C : x - \omega - y - x$ is a cycle in $\mathfrak{ag}(\mathcal{L})$ of length three and no edge of C is an edge of $\Gamma(\mathcal{L})$. □

Corollary 2.9. *Let* $\mathcal{L}$ *be a lattice with bottom element* 0 *such that* $\mathfrak{ag}(\mathcal{L}) \neq \Gamma(\mathcal{L})$. *Then* $\mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = 3$. *Furthermore, there is a cycle* C *of length three in* $\mathfrak{ag}(\mathcal{L})$ *such that no edge of* C *is an edge of* $\Gamma(\mathcal{L})$.

A lattice $\mathcal{L}$ is called *directly indecomposable* if $\mathcal{L}$ has no representation in the form $\mathcal{L} = A \times B$, where both A and B have more than one element. It has been shown in [14, Proposition 2.5] that for a finite commutative ring R with identity if the annihilator graph $AG(R)$ is regular, then $R \cong \mathbb{F} \times \mathbb{F}$, where $\mathbb{F}$ is a field, or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is a local ring. In Theorem 2.11, we determine some conditions on a finite lattice $\mathcal{L}$ under which the annihilator graph $\mathfrak{ag}(\mathcal{L})$ is regular.

Theorem 2.10. [18, Lamma 278] *Let* $\mathcal{L}$ *be a bounded lattice. If* $\mathcal{L}$ *is of finite length, then* $\mathcal{L}$ *is isomorphic to a direct product of directly indecomposable lattices.*

Theorem 2.11. *Let* $\mathcal{L} \cong \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$ *be a finite lattice such that every* $\mathcal{L}_j$ *is a directly indecomposable component of* $\mathcal{L}$ *and* $|\mathcal{L}| \geq 2$. *If* $\mathfrak{ag}(\mathcal{L})$ *is a regular graph and* $\Pi_{j \neq i} |\mathcal{L}_j| \nmid |Z(\mathcal{L}_i)^\star|$ *for all* $i \in \{1, \ldots, n\}$, *then* $\mathcal{L}$ *is a product of at most two directly indecomposable lattices or* $\mathcal{L} \cong C_2 \times C_2 \times C_2$ *where* C_2 *is the two-element chain.*

Proof. We consider different cases about n, the number of directly indecomposable components of $\mathcal{L}$. If $n = 1$, then $\mathcal{L}$ is directly indecomposable. So we assume that $n \geq 2$. If at least one $\mathcal{L}_i$ contains a nonzero zero-divisor, without loss of generality let $\mathcal{L}_1$ contains $y_1 \in Z(\mathcal{L}_1)^\star \cap \mathcal{A}(\mathcal{L}_1)$. Now if we consider $e_1 = (1, 0, \ldots, 0)$ and $y = (y_1, 0, \ldots, 0)$, then $(t_1, \ldots, t_n) \in N_{\mathfrak{ag}(\mathcal{L})}(e_1)$ if and only if $\mathrm{ann}_{\mathcal{L}}(t_1, 0, \ldots, 0) \neq \mathrm{ann}_{\mathcal{L}}(t_1, \ldots, t_n) \cup \mathrm{ann}_{\mathcal{L}}(e_1)$, and this is in turn equivalent to say that $(t_1, \ldots, t_n) \in$

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$Z(\mathcal{L}_1)\times\mathcal{L}_2\times\cdots\times\mathcal{L}_n\setminus(Z(\mathcal{L}_1)\times\{0\}\times\cdots\times\{0\})$. Therefore $\deg_{\mathfrak{ag}(\mathcal{L})}(e_1) = |Z(\mathcal{L}_1)|(|\mathcal{L}_2|\cdots|\mathcal{L}_n|-1)$. Since y_1 is an atom, with a similar argument it can be proved that $\deg_{\mathfrak{ag}(\mathcal{L})}(y) = |Z(\mathcal{L}_1)\setminus[y_1)||\mathcal{L}_2|\cdots|\mathcal{L}_n|-1$, where $[y_1)$ is the filter generated by y_1. Now using the assumption $\Pi_{j=2}^{n}|\mathcal{L}_j| \nmid |Z(\mathcal{L}_1)^\star|$, implies that $\deg_{\mathfrak{ag}(\mathcal{L})}(e_1) \neq \deg_{\mathfrak{ag}(\mathcal{L})}(y)$ which contradicts the regularity of $\mathfrak{ag}(\mathcal{L})$. We assume $n \geq 2$ and $Z(\mathcal{L}_i) = \{0\}$, for each $i = 1,\ldots,n$. Therefore $N_{\mathfrak{ag}(\mathcal{L})}(e_1) = \{y \in Z(\mathcal{L})^\star |\, e_1 \wedge y = 0\}$ and $\deg_{\mathfrak{ag}(\mathcal{L})}(e_1) = |\mathcal{L}_2|\cdots|\mathcal{L}_n| - 1$. With a similar argument for every $i = 1,\ldots,n$, if we take $e_i = (0,\ldots,0,1,0,\ldots,0)$, where 1 occures only in the i^{th} component and every other component is zero, then $\deg_{\mathfrak{ag}(\mathcal{L})}(e_i) = |\mathcal{L}_1|\cdots|\mathcal{L}_{i-1}||\mathcal{L}_{i+1}|\cdots|\mathcal{L}_n| - 1$. By regularity of $\mathfrak{ag}(\mathcal{L})$, we obtain $|\mathcal{L}_1| = \cdots = |\mathcal{L}_n| := t$.

In case $n \geq 3$, let $z = (1,1,0,\ldots,0)$. Then $\deg_{\mathfrak{ag}(\mathcal{L})}(z) = (t^{(n-2)} - 1) + 2(t-1)(t^{(n-2)} - 1)$. Now if $n \geq 4$, it can be easily seen that $\deg_{\mathfrak{ag}(\mathcal{L})}(e_1) \gneqq \deg_{\mathfrak{ag}(\mathcal{L})}(z)$ which contradicts the regularity of the graph. In case $n = 3$, the equality $\deg_{\mathfrak{ag}(\mathcal{L})}(e_1) = \deg_{\mathfrak{ag}(\mathcal{L})}(z)$ implies that $|\mathcal{L}_1| = t = 2$. This means that $\mathcal{L} \cong C_2 \times C_2 \times C_2$. In case $n = 2$, $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L}) = K^{t-1,t-1}$ and $\mathcal{L} \cong \mathcal{L}_1 \times \mathcal{L}_1$. □

It has been shown in [14, Proposition 2.6] that for a finite commutative ring R, the domination number of the annihilator graph $AG(R)$ is at most 2. In the following theorem, we show that a similar (generalized) assertion is true for the annihilator graph of a bounded lattice $\mathcal{L}$ such that it is a direct product of two lattices.

Theorem 2.12. *Let $\mathcal{L}$ be a bounded lattice which is a direct product of two lattices. Then* $\mathrm{dt}(\mathfrak{ag}(\mathcal{L})) \leq 2$.

Proof. Let $\mathcal{L} \cong \mathcal{L}_1 \times \mathcal{L}_2$, for some lattices $\mathcal{L}_1$ and $\mathcal{L}_2$. Now let $A_1 = \{(x_1,0)|x_1 \in \mathcal{L}_1^\star\}$, $A_2 = \{(0,x_2)|x_2 \in \mathcal{L}_2^\star\}$, $C_1 = \{(x_1,x_2)|x_1 \in Z(\mathcal{L}_1)^\star, x_2 \in \mathcal{L}_2^\star\}$ and $C_2 = \{(x_1,x_2)|x_1 \in \mathcal{L}_1^\star, x_2 \in Z(\mathcal{L}_2)^\star\}$. Then $Z(\mathcal{L})^\star = A_1 \cup A_2 \cup C_1 \cup C_2$. We claim that $D = \{x = (1,0), y = (0,1)\}$ is a dominating set of $\mathfrak{ag}(\mathcal{L})$. Let $z = (z_1,z_2) \in Z(\mathcal{L})^\star \setminus D$. If z is an element of A_1 or A_2, then z is adjacent to x or y. Therefore without loss of generality assume that $z \in C_1$. Then $\mathrm{ann}_{\mathcal{L}}(x \wedge z) = \mathrm{ann}_{\mathcal{L}}(z_1,0) = A_2 \cup \{(q,t)|q \in \mathrm{ann}_{\mathcal{L}_1}(z_1), t \in \mathcal{L}_2\}$. Moreover, $\mathrm{ann}_{\mathcal{L}}(x) = A_2 \cup \{(0,0)\}$ and if $z_2 \notin Z(\mathcal{L}_2)$, then $\mathrm{ann}_{\mathcal{L}}(z) = \{(q,0)|q \in \mathrm{ann}_{\mathcal{L}_1}(z_1)\}$ and otherwise $\mathrm{ann}_{\mathcal{L}}(z) = \{(q_1,q_2)|q_1 \in \mathrm{ann}_{\mathcal{L}_1}(z_1), q_2 \in \mathrm{ann}_{\mathcal{L}_2}(z_2)\}$. Thus z is adjacent to x and we conclude that D is a dominating set of $\mathfrak{ag}(\mathcal{L})$. □

We recall that a lattice $\mathcal{L}$ is *of finite length* n if there is a chain in $\mathcal{L}$ of length n and all chains in $\mathcal{L}$ are of length $\leqq n$. A lattice $\mathcal{L}$ is of finite length if it is of length n for some natural number n. A lattice $\mathcal{L}$ is called *atomic* if for every nonzero element $a \in \mathcal{L}$, there exists an atom $e \in \mathcal{L}$ such that $e \leq a$. Every lattice which satisfies the descending chain condition is atomic. In particular, every lattice of finite length is atomic (see [18, Exercises 6.18, 6.19 and 6.20]).

Proposition 2.13. *Let $\mathcal{L}$ be a bounded lattice of finite length. Then there exists a vertex in $\mathfrak{ag}(\mathcal{L})$ which is adjacent to all other vertices of $\mathfrak{ag}(\mathcal{L})$ if and only if either $\mathcal{L} \cong C_2 \times \mathcal{L}'$, where $\mathcal{L}'$ is a directly*

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indecomposable lattice with only one atom or $\mathcal{L}$ is a directly indecomposable with an atom which is adjacent to all other vertices of $\mathfrak{ag}(\mathcal{L})$.

Proof. By Theorem 2.10, $\mathcal{L} \cong \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$ such that for each i with $1 \leq i \leq n$, $\mathcal{L}_i$ is a directly indecomposable lattice. Suppose $x = (x_1, \ldots, x_n) \in Z(\mathcal{L})^\star$ is a vertex which is adjacent to all other vertices of $\mathfrak{ag}(\mathcal{L})$. First we assume that $n \geq 3$. Let $x' = (0, \ldots, 0, x_i, 0, \ldots, 0)$ where x_i is a nonzero component of x in the i^{th} position. Then $\mathrm{ann}_{\mathcal{L}}(x \wedge x') = \mathrm{ann}_{\mathcal{L}}(x')$. This yields $x = x'$. Now let $x'' = (0, \ldots, 0, 1, x_i, 0, \ldots, 0) \in Z(\mathcal{L})^\star$. Then $\mathrm{ann}_{\mathcal{L}}(x \wedge x'') = \mathrm{ann}_{\mathcal{L}}(x)$, which implies that $x' = x = x''$. From this contradiction, we must have $n \leq 2$. If $n = 1$ and $|\mathcal{A}(\mathcal{L})| = 1$, then $\mathfrak{ag}(\mathcal{L})$ is an empty graph. Otherwise, if $|\mathcal{A}(\mathcal{L})| \geq 2$, then $|Z(\mathcal{L})^\star| \gneqq 1$. In this case for each $l \in Z(\mathcal{L})^\star \setminus \mathcal{A}(\mathcal{L})$ there is an atom e where $e \lneqq l$ and $\mathrm{ann}_{\mathcal{L}}(e \wedge l) = \mathrm{ann}_{\mathcal{L}}(e)$. Hence x must be an atom which is adjacent to all other vertices. Now for $n = 2$, $\mathcal{L} \cong \mathcal{L}_1 \times \mathcal{L}_2$, let $x = (x_1, x_2) \in Z(\mathcal{L})^\star$, $x_1 \neq 0$ and $x_2 \neq 0$. Since $(x_1, 0) \in Z(\mathcal{L})^\star$ is not adjacent to x, we must have $x = (x_1, 0)$ and it can be easily shown that $x_1 \in \mathcal{L}_1$ is the only atom in $\mathcal{L}_1$. Also for each $l \neq x_1$ of $\mathcal{L}_1$, $(l, 0) \in Z(\mathcal{L})^\star$ is not adjacent to x. Therefore, $\mathcal{L}_1$ is a directly indecomposable lattice isomorphic to C_2. Now if $\mathcal{L}_2$ contains at least two atoms e_1' and e_2', then $(x_1, e_1') \in Z(\mathcal{L})^\star$ is not adjacent to x. This means that the statement holds. The "if" portion follows trivially. □

Corollary 2.14. *Let $\mathcal{L}$ be a bounded lattice of finite length and $\mathfrak{ag}(\mathcal{L}) \neq \emptyset$. Then $\mathfrak{ag}(\mathcal{L})$ is a complete graph if and only if $\mathcal{L} \cong C_2 \times C_2$ or $Z(\mathcal{L})^\star = \mathcal{A}(\mathcal{L})$.*

Proof. If $\mathfrak{ag}(\mathcal{L})$ is a complete graph by Proposition 2.13, we have the following two cases:

Case(1): $\mathcal{L} \cong C_2 \times \mathcal{L}'$ such that $\mathcal{L}'$ is a directly indecomposable lattice with only one atom e_1'. Now since for each $l' \in \mathcal{L}'$ which is not an atom, the vertex $(0, l')$ is not adjacent to $(0, e_1')$, it can be concluded that $\mathcal{L}' \cong C_2$.

Case(2): $\mathcal{L}$ is a directly indecomposable and there is an atom $e \in \mathcal{L}$ which is adjacent to every other vertex. Whereas $|\mathcal{A}(\mathcal{L})| \geq 2$, assume on contrary that $\mathcal{A}(\mathcal{L}) \subsetneq Z(\mathcal{L})^\star$ then there exists $l \in Z(\mathcal{L})^\star \setminus \mathcal{A}(\mathcal{L})$ such that $l \wedge e = 0$ for each atom of $\mathcal{L}$ this is a contradiction, since $\mathcal{L}$ is atomic. □

3. Equality of $\mathfrak{ag}(\mathcal{L})$ and $\Gamma(\mathcal{L})$

We recall that a lattice $\mathcal{L}$ is called a *diamond* if every element of $\mathcal{L} \setminus \{0, 1\}$ is both an atom and a coatom. By [15, Proposition 3.4], the zero-divisor graph of a diamond $\mathcal{D}$ is a complete graph. Therefore it is also identical to its annihilator graph $\mathfrak{ag}(\mathcal{D})$. In this section, for a distributive lattice $\mathcal{L}$ with 0, we investigate some conditions under which the graphs $\mathfrak{ag}(\mathcal{L})$ and $\Gamma(\mathcal{L})$ are identical.

At first we recall some results about $\Gamma(\mathcal{L})$ which we will use latter.

Theorem 3.1. [25, Theorem 2.2]. *Let $\mathcal{L}$ be a distributive lattice with the least element 0. Then $\Gamma(\mathcal{L})$ is a complete bipartite graph if and only if there exist prime ideals P_1 and P_2 in $\mathcal{L}$ such that $P_1 \cap P_2 = \{0\}$.*

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Theorem 3.2. [21, Theorem 2.11]. *Let $\mathcal{L}$ be a lattice with 0. Then the following statements are equivalent.*

(1) *The zero-divisor graph $\Gamma(\mathcal{L})$ is a complete bipartite graph.*
(2) *The zero-divisor graph $\Gamma(\mathcal{L})$ is a bipartite graph.*

Theorem 3.3. [21, Theorem 2.27]. *Let $\mathcal{L}$ be a 0-distributive lattice and $Z(\mathcal{L}) = V(\Gamma(\mathcal{L})) \cup \{0\}$ is not an ideal. Then $\operatorname{diam}(\Gamma(\mathcal{L})) \leq 2$ if and only if $\mathcal{L}$ has exactly two minimal prime ideals.*

Theorem 3.4. [3, Theorem 4.2]. *Let P be a poset. Then the following assertions hold*:

(1) $\operatorname{gr}(\Gamma(P)) \in \{3, 4, \infty\}$.
(2) $\operatorname{gr}(\Gamma(P)) = \infty$ *if and only if* $\Gamma(P)$ *is a star graph.*
(3) $\operatorname{gr}(\Gamma(P)) = 4$ *if and only if* $\Gamma(P)$ *is a bipartite but not a star graph.*
(4) $\operatorname{gr}(\Gamma(P)) = 3$ *if and only if* $\Gamma(P)$ *contains an odd cycle.*

Lemma 3.5. *Let $\mathcal{L}$ be a lattice with $Z(\mathcal{L}) \neq \{0\}$. Then*:

If $c \vee z \in Z(\mathcal{L})$ with $c \in \operatorname{ann}_{\mathcal{L}}(z) \setminus \{0\}$, then $\operatorname{ann}_{\mathcal{L}}(c \vee z) \subsetneq \operatorname{ann}_{\mathcal{L}}(z)$. In particular if $Z(\mathcal{L})$ is an ideal of $\mathcal{L}$ and $c \in \operatorname{ann}_{\mathcal{L}}(z) \setminus \{0\}$, then $\operatorname{ann}_{\mathcal{L}}(c \vee z)$ is properly contained in $\operatorname{ann}_{\mathcal{L}}(z)$.

Proof. Since $c \leq (c \vee z)$, $c \vee z \neq 0$ and $c \notin \operatorname{ann}_{\mathcal{L}}(c \vee z)$. Hence $\operatorname{ann}_{\mathcal{L}}(c \vee z) \neq \operatorname{ann}_{\mathcal{L}}(z)$ and since $z \leq c \vee z$ it follows that $\operatorname{ann}_{\mathcal{L}}(c \vee z) \subsetneq \operatorname{ann}_{\mathcal{L}}(z)$. □

Theorem 3.6. *Let $\mathcal{L}$ be a lattice such that $Z(\mathcal{L}) \neq \{0\}$. Then the following assertions are equivalent.*

(1) $\mathfrak{ag}(\mathcal{L})$ *is complete;*
(2) $\Gamma(\mathcal{L})$ *is complete.*

Proof. (1) ⇒ (2). Let x and y be distinct elements in $Z(\mathcal{L})^\star$ which are adjacent in $\mathfrak{ag}(\mathcal{L})$. We show that $x - y$ is an edge of $\Gamma(\mathcal{L})$. Suppose on contrary that $x \wedge y \neq 0$. Since $x - y$ is an edge of $\mathfrak{ag}(\mathcal{L})$, we have $\operatorname{ann}_{\mathcal{L}}(x \wedge y) \neq \operatorname{ann}_{\mathcal{L}}(x)$, and thus $x \wedge y \neq x$. Since we have $\operatorname{ann}_{\mathcal{L}}(x \wedge (x \wedge y)) = \operatorname{ann}_{\mathcal{L}}(x \wedge y)$, therefore x is not adjecent to $x \wedge y$ in $\mathfrak{ag}(\mathcal{L})$ which contradicts the completeness of $\mathfrak{ag}(\mathcal{L})$. Hence $x \wedge y = 0$ and $x - y$ is an edge of $\Gamma(\mathcal{L})$.

(2) ⇒ (1) is trivial. □

Remark 3.7. *As we saw in Theorem 3.6, the completness condition for $\mathfrak{ag}(\mathcal{L})$ is the same as that of $\Gamma(\mathcal{L})$. However, it can not concluded that each complete subgraph of $\mathfrak{ag}(\mathcal{L})$ is a complete subgraph of $\Gamma(\mathcal{L})$. For example, if we take $\mathcal{L} = \mathcal{D}(210)$, the lattice of all natural divisors of 210, then $\{6, 10, 15, 14, 21, 35\}$ is a clique in $\mathfrak{ag}(\mathcal{L})$, while the largest clique in $\Gamma(\mathcal{L})$ has only 4 elements (see also Theorem 4.3 part (4)).*

The set of all maximal ideals of $\mathcal{L}$ is denoted by $\operatorname{Max}(\mathcal{L})$. A lattice $\mathcal{L}$ with the bottom element 0 is called a *semiprimitive lattice* if $\bigcap \operatorname{Max}(\mathcal{L}) = \{0\}$.

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Following Y.S. Pawar and N.K. Thakare [24], we call a bounded distributive lattice $\mathcal{L}$ a *pm-lattice* (or a *Gelfand lattice*) if every prime ideal in $\mathcal{L}$ is contained in a unique maximal ideal. By [20, Theorem 2.11], for a semiprimitive pm-lattice $\mathcal{L}$ with $|\mathcal{L}| \geq 5$, the diameter $\Gamma(\mathcal{L})$ is $\min\{|\mathrm{Max}(\mathcal{L})|, 3\}$. This yields by Corollary 2.5 and Theorem 3.6:

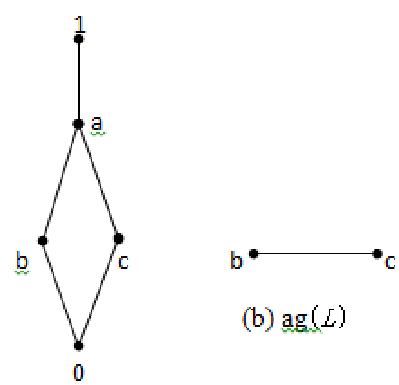
Corollary 3.8. *Let $\mathcal{L}$ be a semiprimitive pm-lattice with $Z(\mathcal{L}) \neq \{0\}$ and $|\mathcal{L}| \geq 5$. Then* $\mathrm{diam}(\mathfrak{ag}(\mathcal{L})) = min\{|\mathrm{Max}(\mathcal{L})|, 2\}$.

Theorem 3.9. *Let $\mathcal{L}$ be a distributive lattice with 0 such that $Z(\mathcal{L}) \neq \{0\}$ and $Z(\mathcal{L})$ is an ideal of $\mathcal{L}$. Then $\mathfrak{ag}(\mathcal{L}) \neq \Gamma(\mathcal{L})$ and $\mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = 3$.*

Proof. Let $z \in Z(\mathcal{L})^\star, c \in \mathrm{ann}_{\mathcal{L}}(z) \setminus \{0\}$ and $m \in \mathrm{ann}_{\mathcal{L}}(c \vee z) \setminus \{0\}$. Then $m \in \mathrm{ann}_{\mathcal{L}}(c \vee z) \subsetneq \mathrm{ann}_{\mathcal{L}}(c)$ and $m \wedge c = 0$. Since $c \neq 0$, we have $m \neq c$ and since $c \wedge (m \vee z) = 0$ and $c \wedge (c \vee z) = c \neq 0$, hence $c \vee z \neq m \vee z$. Now since $(m \vee z) \wedge (c \vee z) = z \vee (m \wedge c) = z \neq 0$, we have $(c \vee z)$ and $(m \vee z)$ are not adjacent in $\Gamma(\mathcal{L})$. Since $c \neq 0, m \neq 0$ it follows that $(c \vee m) \in \mathrm{ann}_{\mathcal{L}}(z) \setminus (\mathrm{ann}_{\mathcal{L}}(c \vee z) \cup \mathrm{ann}_{\mathcal{L}}(m \vee z))$ and thus $(c \vee z) - (m \vee z)$ is an edge of $\mathfrak{ag}(\mathcal{L})$. Therefore $\mathfrak{ag}(\mathcal{L}) \neq \Gamma(\mathcal{L})$ and $\mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = 3$ by Corollary 2.9. □

The following example shows that the assertion in the Theorem 3.9 does not hold if $Z(\mathcal{L})$ is not an ideal.

Example 3.10. *Let $\mathcal{L} = \{0, a, b, c, 1\}$ be the lattice with $0 \prec b \prec a$, $0 \prec c \prec a$ and $b \| c$. Then $Z(\mathcal{L}) = \{0, b, c\}$ is not an ideal and $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$.*



(a) $\mathcal{L}$

(b) $\mathfrak{ag}(\mathcal{L})$

Fig.2.

Theorem 3.11. *Let $\mathcal{L}$ be a distributive lattice with zero, $Z(\mathcal{L}) \neq \{0\}$ and $|\mathrm{Min}(\mathcal{L})| \geq 3$. Then $\mathfrak{ag}(\mathcal{L}) \neq \Gamma(\mathcal{L})$ and $\mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = 3$.*

Proof. If $Z(\mathcal{L})$ is an ideal of $\mathcal{L}$, then $\mathfrak{ag}(\mathcal{L}) \neq \Gamma(\mathcal{L})$ by Theorem 3.9. Now assume that $Z(\mathcal{L})$ is not an ideal of $\mathcal{L}$. By Theorem 3.3, $\mathrm{diam}(\Gamma(\mathcal{L})) = 3$. By Corollary 2.5 we have $\mathfrak{ag}(\mathcal{L}) \neq \Gamma(\mathcal{L})$ and by Corollary 2.9, $\mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = 3$. □

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Theorem 3.12. *Let $\mathcal{L}$ be a distributive lattice with zero, $Z(\mathcal{L}) \neq \{0\}$. Then $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$ if and only if $|\mathrm{Min}(\mathcal{L})| = 2$.*

Proof. Let $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$ and $Z(\mathcal{L}) \neq \{0\}$. By Theorem 3.9, $Z(\mathcal{L})$ is not an ideal and by Theorem 3.3, $\mathcal{L}$ has exactly two minimal prime ideals. Conversely, suppose that $|\mathrm{Min}(\mathcal{L})| = 2$ and let P_1 and P_2 be the minimal prime ideals of $\mathcal{L}$. By [20, Corollary 1.8] and [18, Exercise 1.34, Page 124], $P_1 \cap P_2 = \{0\}$ and then $Z(\mathcal{L}) = P_1 \cup P_2$. Now let $a - b$ be an edge in $\mathfrak{ag}(\mathcal{L})$. If $a, b \in P_1$, since $P_1 \cap P_2 = \{0\}$, thus $a \wedge b \neq 0$ and it follows that $\mathrm{ann}_{\mathcal{L}}(a \wedge b) = \mathrm{ann}_{\mathcal{L}}(a) = \mathrm{ann}_{\mathcal{L}}(b) = P_2$. Therefore a and b are not adjacent in $\mathfrak{ag}(\mathcal{L})$, a contradiction. Similarly, the assumption $a, b \in P_2$ yields the same contradiction. The only remained possibility is $a \in P_1, b \in P_2$. In this case $a \wedge b = 0$. This means that $a - b$ is an edge of $\Gamma(\mathcal{L})$ and therefore $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$. □

Let $\mathcal{L}$ be a distributive lattice with the bottom element 0. It has been shown in [20, Theorem 2.18] that if there exists a vertex of $\Gamma(\mathcal{L})$ which is adjacent to every other vertex, then $|\mathrm{Min}(\mathcal{L})| = 2$.

Combining this result with Theorem 3.12, we obtain the following corollary:

Corollary 3.13. *Let $\mathcal{L}$ be a distributive lattice with the bottom element 0 and $Z(\mathcal{L}) \neq \{0\}$. If there exists a vertex of $\Gamma(\mathcal{L})$ which is adjacent to every other vertex, then $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$.*

Theorem 3.14. *Let $\mathcal{L}$ be a distributive lattice with 0. Then the following assertions are equivalent*:

(1) $\mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = 4$;
(2) $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$ *and* $\mathrm{gr}(\Gamma(\mathcal{L})) = 4$;
(3) $\mathrm{gr}(\Gamma(\mathcal{L})) = 4$;
(4) $|\mathrm{Min}(\mathcal{L})| = 2$ *and each minimal prime ideal of $\mathcal{L}$ has at least three distinct elements;*
(5) $\Gamma(\mathcal{L}) = K^{m,n}, \quad m, n \geq 2$;
(6) $\mathfrak{ag}(\mathcal{L}) = K^{m,n}, \quad m, n \geq 2$.

Proof. (1) ⇒ (2). By Corollary 2.9 is trivial.

(2) ⇒ (3). It is obvious.

(3) ⇒ (4). Theorems 3.4 and 3.2 imply that $\Gamma(\mathcal{L})$ is a complete bipartite graph but not a star graph. By Theorem 3.1 there are prime ideals P_1 and P_2 such that $P_1 \cap P_2 = \{0\}$ and $Z(\mathcal{L}) = P_1 \cup P_2$ is not an ideal. Now the diameter of the complete bipartite graph $\Gamma(\mathcal{L})$ is at most 2. By Theorem 3.3 and [18, Exercise 1.34, Page 124], P_1 and P_2 are the only minimal prime ideals in $\mathcal{L}$ and $|P_i| \geq 3$ for $i \in \{1, 2\}$.

(4) ⇒ (5). By [20, Corollary 1.8] and [18, Exercise 1.34, Page 124] and Theorem 3.1 the assertion is clear.

(5) ⇒ (6). By Theorem 3.1, Theorem 3.3 and Theorem 3.12 this statement is explicit.

(6) ⇒ (1). The girth of every complete bipartite graph that is not a star graph equals 4. □

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Theorem 3.15. *Let $\mathcal{L}$ be a distributive lattice with 0. Then the following assertions are equivalent*:

(1) $\mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = \infty$;

(2) $\Gamma(\mathcal{L}) = \mathfrak{ag}(\mathcal{L}), \mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = \infty$;

(3) $\mathrm{gr}(\Gamma(\mathcal{L})) = \infty$;

(4) $|\mathrm{Min}(\mathcal{L})| = 2$ *and at least one minimal prime ideal of* $\mathcal{L}$ *has exactly two distinct elements;*

(5) $\Gamma(\mathcal{L}) = K^{1,n}, \quad n \geq 1$;

(6) $\mathfrak{ag}(\mathcal{L}) = K^{1,n}, \quad n \geq 1$.

Proof. (1) $\Rightarrow$ (2). By Corollary 2.9 that is straightforward to verify.

(2) $\Rightarrow$ (3). No challenges.

(3) $\Rightarrow$ (4). By Theorem 3.4, $\Gamma(\mathcal{L})$ is a star graph and by Theorem 3.1 there are prime ideals P_1 and P_2 such that $Z(\mathcal{L}) = P_1 \cup P_2$ and $P_1 \cap P_2 = \{0\}$. By Theorem 3.3 and hypothesis P_1, P_2 are the only minimal prime ideals in $\mathcal{L}$ such that at least one minimal prime ideal of $\mathcal{L}$ has exactly two distinct elements.

(4) $\Rightarrow$ (5), (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1). Similar to the preceding theorem. □

Corollary 3.16. *Let $\mathcal{L}$ be a distributive lattice with 0. Then* $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$ *if and only if* $\mathrm{gr}(\mathfrak{ag}(\mathcal{L})) = \mathrm{gr}(\Gamma(\mathcal{L})) \in \{4, \infty\}$.

4. The zero-divisor grph and the annihilator graph of the lattice $\mathcal{D}(n)$

Let the natural number n has the prime decomposition $n = p_1^{t_1} \cdots p_k^{t_k}$ with $t_i \in \mathbb{N}$ and $k \geq 2$. Then $\mathcal{L} := (\mathcal{D}(n), |)$ is a bounded distributive lattice with the bottom element 1. For every $x, y \in \mathcal{D}(n)$ the meet $x \wedge y$ is the greatest common divisor of x and y and the join $x \vee y$ is the least common multiple of them. In this section, we consider the zero-divisor graph $\Gamma(\mathcal{L})$ and the annihilator graph $\mathfrak{ag}(\mathcal{L})$ associated to this lattice. We investigate some properties and invariants of these graphs. Among other results, we compute the domination number, the clique number and the chromatic number of these graphs. Also, we determine some cases in which these graphs are planar, Eulerian or Hamiltonian. The vertex set in both graphs is

$$Z(\mathcal{L}) \setminus \{1\} = \{x \in \mathcal{D}(n) | \gcd(x, y) = 1, \text{for some } y \in \mathcal{D}(n) \setminus \{1\}\}.$$

In $\Gamma(\mathcal{L})$, two distinct vertices x and y are adjacent if and only if $\gcd(x, y) = 1$. The following lemma gives a simple criterion for adjacence of vertices in $\mathfrak{ag}(\mathcal{D}(n))$.

Lemma 4.1. *Let $n \geq 4$ be a non-prime natural number and $\mathcal{L} := (\mathcal{D}(n), |)$ be the lattice of all natural divisors of n. Then two vertices x and y are adjacent in $\mathfrak{ag}(\mathcal{L})$ if and only if x has a prime divisor p which is not a divisor of y and also y has a prime divisor q which is not a divisor of x.*

Proof. (⇒). Let x and y be two adjacent vertices in $\mathfrak{ag}(\mathcal{L})$ and $\gcd(x,y)=d$. $\mathrm{ann}_{\mathcal{L}}(d)\neq \mathrm{ann}_{\mathcal{L}}(x)\cup \mathrm{ann}_{\mathcal{L}}(y)$ gives a natural number t with $\gcd(t,d)=1$, $\gcd(t,x)=x^{'}\neq 1$ and $\gcd(t,y)=y^{'}\neq 1$. Therefore, there exists a prime divisor p of $x^{'}$ with $p\nmid y$. Also, there exists a prime divisor q of $y^{'}$ with $q\nmid x$.

(⇐). Let p be a prime divisor of x such that $p\nmid y$ and q be a prime divisor of y such that $q\nmid x$. If $\gcd(x,y)=d$, then $\gcd(pq,d)=1$, $\gcd(pq,x)=p\neq 1$ and $\gcd(pq,y)=q\neq 1$. This means that $pq\in \mathrm{ann}_{\mathcal{L}}(d)\setminus(\mathrm{ann}_{\mathcal{L}}(x)\cup \mathrm{ann}_{\mathcal{L}}(y))$. □

We recall that a graph G is said to be *planar* if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex. A graph is *Hamiltonian* if it has a *spanning cycle*. A connected graph is *Eulerian* if it has a closed trail containing all edges.

Lemma 4.2. [27, Theorem 6B] *A conneced graph G is Eulerian if and only if every vertex of G has an even degree.*

Theorem 4.3. *Let n be a natural number and $\mathcal{L}=\mathcal{D}(n)$ be the lattice containing all natural divisors of n. If n has the prime decomposition $n=p_1^{t_1}\cdots p_k^{t_k}$, with $1\leq t_1\leq\cdots\leq t_k$ and $k\geq 2$, then*

(1) *The graphs $\mathfrak{ag}(\mathcal{L})$ and $\Gamma(\mathcal{L})$ are identical if and only if $k=2$.*
(2) *If $k=2$ and $t_1=1$, then $\mathrm{dt}(\mathfrak{ag}(\mathcal{L}))=\mathrm{dt}(\Gamma(\mathcal{L}))=1$.*
(3) *If $n\neq p_1p_2^{t_2}$, then:*
  (*i*) $\mathrm{dt}(\Gamma(\mathcal{L}))=k$.
  (*ii*) $\mathrm{dt}(\mathfrak{ag}(\mathcal{L}))=2$.
(4) $\omega(\Gamma(\mathcal{L}))=\chi(\Gamma(\mathcal{L}))=k$.
(5) $\omega(\mathfrak{ag}(\mathcal{L}))=\chi(\mathfrak{ag}(\mathcal{L}))$ *equals* $\binom{k}{k/2}$ *or* $\binom{k}{(k-1)/2}$, *if k is an even or odd number respectively.*
(6) *If $k\geq 4$, then none of $\mathfrak{ag}(\mathcal{L})$ or $\Gamma(\mathcal{L})$ is planar.*
(7) *If $k=3$, then $\mathfrak{ag}(\mathcal{L})$ is planar if and only if $t_1=t_2=t_3=1$.*
(8) *If $k=3$, then $\Gamma(\mathcal{L})$ is planar if and only if $t_1=t_2=1, t_3=2$ or $t_1=t_2=t_3=1$.*
(9) *If $k=2$, then $\mathfrak{ag}(\mathcal{L})=\Gamma(\mathcal{L})$ is planar if and only if $t_1\leq 2$.*
(10) *The case $n=p_1p_2p_3^2$ is the only case in which $\Gamma(\mathcal{L})$ is planar but $\mathfrak{ag}(\mathcal{L})$ is not.*
(11) *$\mathfrak{ag}(\mathcal{L})$ is Eulerian if and only if for each $1\leq i\leq k$, t_i is even.*
(12) *$\Gamma(\mathcal{L})$ is Eulerian if and only if for each $1\leq i\leq k$, t_i is even.*
(13) *$\Gamma(\mathcal{L})$ is Hamiltonian if and only if $k=2$ and $t_1=t_2\geq 2$.*
(14) *Let for every i with $2\leq i\leq k$ yield $t_i\cdots t_k\leq\sum(t_{j_1}\cdots t_{j_{k-i+1}})$ where the sum varies over all products of (k-i+1)-tuples of exponents of prime factors of n except $t_i\cdots t_k$. Then $\mathfrak{ag}(\mathcal{L})$ is Hamiltonian.*

Proof. (1) If $k=2$, then $\mathfrak{ag}(\mathcal{L})=\Gamma(\mathcal{L})=K^{t_1,t_2}$. Now let $k\geq 3$. Then $x=p_1p_2$ and $y=p_1p_3$ are adjacent in $\mathfrak{ag}(\mathcal{L})$ but not in $\Gamma(\mathcal{L})$. Therefore, $\mathfrak{ag}(\mathcal{L})\neq\Gamma(\mathcal{L})$.

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(2) Let $n=p_1p_2^{t_2}$. Then $\mathfrak{ag}(\mathcal{L})=\Gamma(\mathcal{L})$, by part (1). By Theorem 2.12, $\mathrm{dt}(\mathfrak{ag}(\mathcal{L}))=\mathrm{dt}(\Gamma(\mathcal{L}))\leq 2$. Also, it can be easily shown that if $n=p_1p_2^{t_2}$ then $\Gamma(\mathcal{L})$ is a star graph and $\mathrm{dt}(\Gamma(\mathcal{L}))=1$.

(3) (i) For $k\geq 3$, let $A_i=\{p_{i_1}^{\alpha_{i_1}}\cdots p_{i_{k-1}}^{\alpha_{i_{k-1}}}|i_j\neq i, 1\leq j\leq k-1, 1\leq\alpha_{i_j}\leq t_{i_j}\}\cup\{p_i^{\alpha_i}|1\leq\alpha_i\leq t_i\}$. Since every element in the first part of A_i is only adjacent to elements in the second part of A_i, every dominating set in $\Gamma(\mathcal{L})$ must have at least one element in A_i. Thus $\mathrm{dt}(\Gamma(\mathcal{L}))\geq k$. But it is easy to see that $\{p_1,\ldots,p_k\}$ is also a dominating set. This means that $\mathrm{dt}(\Gamma(\mathcal{L}))=k$. Now for $k=2$, if $n=p_1^{t_1}p_2^{t_2}$ with $t_1\neq 1$, then $\Gamma(\mathcal{L})$ is a complete bipartite graph and $\mathrm{dt}(\Gamma(\mathcal{L}))=2$.

(ii) For $\mathfrak{ag}(\mathcal{L})$ in case $k=2$, we have $\mathfrak{ag}(\mathcal{L})=\Gamma(\mathcal{L})$ and the domination number computed above. In case $k\geq 3$, according to the Theorem 2.12, $\mathrm{dt}(\mathfrak{ag}(\mathcal{L}))\leq 2$. We show that $\mathrm{dt}(\mathfrak{ag}(\mathcal{L}))\neq 1$. If $\mathcal{D}=\{x\}$ is a dominating set, then $x=p_i^{\alpha_i}$ with $1\leq\alpha_i\leq t_i$ or $x=p_{i_1}^{\alpha_{i_1}}\cdots p_{i_s}^{\alpha_{i_s}}$, with $s\geq 2$. In the first case , xp_j is not adjacent to x for every $p_j\neq p_i$. In the last case, p_{i_1} is not adjacent to x. Therefore in any case $\mathrm{dt}(\mathfrak{ag}(\mathcal{L}))=2$.

(4) By [15, Theorem 5.13], the clique number of $\Gamma(\mathcal{L})$ is the number of atoms in $\mathcal{L}$ which is equal to k.Whereas $k=\omega(\Gamma(\mathcal{L}))\leqslant\chi(\Gamma(\mathcal{L}))$ and vertices of graph can be partition into k mutually disjoint parts each of which containing the multiples of p_i which are not multiples of p_j for any $j\leq i-1$. Then $\chi(\Gamma(\mathcal{L}))\leq k$ and therefore $\omega(\Gamma(\mathcal{L}))=\chi(\Gamma(\mathcal{L}))=k$.

(5) For obtaining the clique number and the chromatic number of $\mathfrak{ag}(\mathcal{L})$, we note that for every natural number t with $1\leq t\leq k-1$, the set

$$\{\prod_{i=1}^{t}q_i|q_i\in\{p_1,\ldots,p_k\}\}$$

is a clique in $\mathfrak{ag}(\mathcal{L})$ of cardinality $\binom{k}{t}$. Now if k is an even number, the largest clique of this kind is of cardinality $\binom{k}{k/2}$. This means that $\binom{k}{k/2}\leq\omega(\mathfrak{ag}(\mathcal{L}))$. On the other hand, we can partition the vertex set of $\mathfrak{ag}(\mathcal{L})$ into $\binom{k}{k/2}$ seperated parts such that no two distinct vertices in any part are adjacent. In other words $\mathfrak{ag}(\mathcal{L})$ is a $\binom{k}{k/2}$-partite graph. Therefore the chromatic number $\chi(\mathfrak{ag}(\mathcal{L}))$ is at most $\binom{k}{k/2}$. Consequently, $\omega(\mathfrak{ag}(\mathcal{L}))=\chi(\mathfrak{ag}(\mathcal{L}))=\binom{k}{k/2}$. In case k is an odd number, a similar argument shows that $\omega(\mathfrak{ag}(\mathcal{L}))=\chi(\mathfrak{ag}(\mathcal{L}))=\binom{k}{(k-1)/2}$.

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(6) when $k \geq 4$, $\Gamma(\mathcal{L})$ has the subgraph:

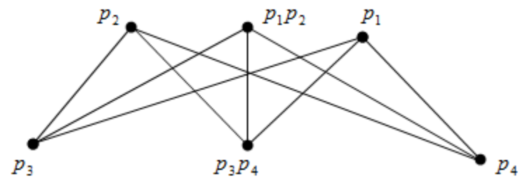
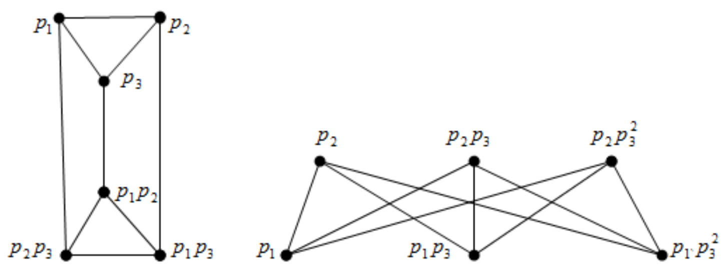


Fig. 3.

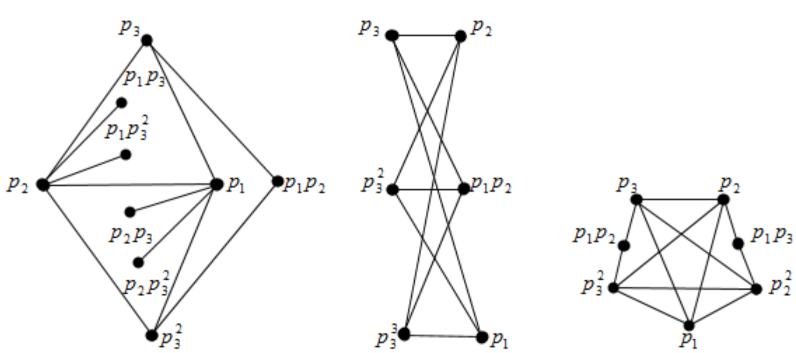
(7) If $k = 3$ and $n = p_1^{t_1}p_2^{t_2}p_3^{t_3}$ with $t_3 \geq 2$ then $\mathfrak{ag}(\mathcal{L})$ has the following complete bipartite graph(**Fig.4**(b)) as a subgraph:



(a) ag(D($p_1p_2p_3$)). (b) A subgraph of ag(D($p_1p_2p_3^2$)).

Fig. 4. The graph ag(D($p_1p_2p_3$)) is planar,but the graph ag(D($p_1p_2p_3^2$)) is not planar.

(8) For $n = p_1p_2p_3^2$, $\Gamma(\mathcal{L})$ is planar, whereas for $n = p_1p_2p_3^3$ and $n = p_1p_2^2p_3^2$, $\Gamma(\mathcal{L})$ has the subgraphs(**Fig.5**(b)) and (**Fig.5**(c)) respectively.



(a) $\Gamma(D(p_1p_2p_3^2))$. (b) A subgraph of $\Gamma(D(p_1p_2p_3^3))$. (c) A subgraph of $\Gamma(D(p_1p_2^2p_3^2))$.

Fig. 5. The graph $\Gamma(D(p_1p_2p_3^2))$ is plnar, but the graphs $\Gamma(D(p_1p_2p_3^3))$ and $\Gamma(D(p_1p_2^2p_3^2))$ are not planar.

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(9) If $n = p_1^{t_1}p_2^{t_2}$ and $t_1 \geq 3$, then $\Gamma(\mathcal{L}) = \mathfrak{ag}(\mathcal{L})$ has the subgraph:

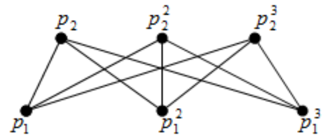

Fig. 6. The graph $\Gamma(D(p_1^3p_2^3))$.

(10) As it can bee seen in parts (7)-(9), the graph $\Gamma(\mathcal{L})$ is planar but $\mathfrak{ag}(\mathcal{L})$ is not if and only if $n = p_1p_2p_3^2$.

(11) We note that for every j with $1 \leq j \leq k-1$, $p_{i_j} \in \{p_1, \ldots, p_k\}$ and $1 \leq \alpha_{i_j} \leq t_{i_j}$, we have

$$\deg_{\mathfrak{ag}(\mathcal{L})}(p_{i_1}^{\alpha_{i_1}} \cdots p_{i_j}^{\alpha_{i_j}}) = \deg_{\mathfrak{ag}(\mathcal{L})}(p_{i_1}^{t_{i_1}} \cdots p_{i_j}^{t_{i_j}}) = [\prod_{s \notin \{i_1,\ldots,i_j\}} (t_s+1) - 1][\prod_{k=1}^{j}(t_{i_k}+1) - \prod_{k=1}^{j} t_{i_k}].$$

Now if each $i \in \{1, \ldots, k\}$, the power t_i is even, then the degree of every vertex is even.

Conversely, assume that $\mathfrak{ag}(\mathcal{L})$ is Eulerian. To the contrary, suppose that there exists an odd exponent t_i. Then for each j with $j \neq i$, $\deg_{\mathfrak{ag}(\mathcal{L})}(p_j) = \prod_{s \neq j}(t_s+1) - 1$ is an odd number. This is a contradiction by Lemma 4.2.

(12) With a similar argument it can be easily seen that

$$\deg_{\Gamma(\mathcal{L})}(p_{i_1}^{\alpha_{i_1}} \cdots p_{i_j}^{\alpha_{i_j}}) = \deg_{\Gamma(\mathcal{L})}(p_{i_1}^{t_{i_1}} \cdots p_{i_j}^{t_{i_j}}) = [\prod_{s \notin \{i_1,\ldots,i_j\}} (t_s+1)] - 1.$$

(13) If $k = 2$ and $t_1 = t_2 \geq 2$, then $\mathfrak{ag}(\mathcal{L}) = \Gamma(\mathcal{L})$ is a complete bipartite graph. Conversely, assume that $\Gamma(\mathcal{L})$ is a Hamiltonian. Suppose, by way of contradiction, that n has at least three factors. Namely let $n = p_1^{t_1}p_2^{t_2}p_3^{t_3}$ and for each$i$ with $1 \leq i \leq 3$, $t_i \neq 1$ (if $t_i = 1$, then $\frac{n}{p_i}$ is a vertex of degree 1 and so in this case $\Gamma(\mathcal{L})$ can not be Hamiltonian). Then $t_1t_2 \leq t_3$, $t_1t_3 \leq t_2$ and $t_2t_3 \leq t_1$ that is imposible. Also, if $t_1 \neq t_2$, we can not obtain a Hamiltonian cycle. .

(14) At first consider

$$\begin{array}{lcl} A_1 & = & \{p_2^{i_2} \cdots p_k^{i_k} \mid 1 \leq i_j \leq t_j, j \neq 1\}, \\ A_2 & = & \{p_1^{i_1}p_3^{i_3} \cdots p_k^{i_k} \mid 1 \leq i_s \leq t_s, s \neq 2\}, \\ & . & \\ & . & \\ & . & \\ A_{k-1} & = & \{p_1^{i_1} \cdots p_{k-2}^{i_{k-2}}p_k^{i_k} \mid 1 \leq i_s \leq t_s, s \neq k-1\}, \\ A_k & = & \{p_1^{i_1} \cdots p_{k-1}^{i_{k-1}} \mid 1 \leq i_s \leq t_s, s \neq k\}. \end{array}$$

It can be easily seen that for every m with $1 \leq m \leq k$, $|A_m| = (t_1 \cdots t_k)/t_m$. In particular by assumption, $|A_1| = t_2 \cdots t_k \leq \sum_{m=2}^{k}(t_1 \cdots t_k)/t_m$. Furthermore, no pair of elements in any A_m are adjacent. However, for $l \neq m$, every element in A_l is adjacent to to all elements in A_m. Now in order to obtain a Hamiltonian cycle, we start with $p_2 \cdots p_k \in A_1$. Using the

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assumption, we can distribute the elements of $\{A_k|2 \leq k \leq m\} \setminus \{p_1 \cdots p_{k-1}\}$ between elements in A_1 and add the element $p_1 \cdots p_{k-1}$ at the end to get a path of all divisors of n with $k-1$ prime factors. As it can be easily seen, the last vertex $p_1 \cdots p_{k-1}$ of this path is itself adjacent to $p_3 \cdots p_k$ with $k-2$ prime factors. Now we can continue this process inductively to obtain a Hamiltonian cycle starting from $p_2 \cdots p_k$ with the end point p_1.

□

Example 4.4. *Let $n = p_1p_2p_3p_4^8$. Then $\mathfrak{ag}(\mathcal{L})$ is not Hamiltonian. However, if $n = p_1p_2p_3p_4^3$. Then $\mathfrak{ag}(\mathcal{L})$ is Hamiltonian and we have the following Hamiltonian cycle.*

$$p_2p_3p_4 - p_1p_3p_4 - p_1p_2p_4 - p_2p_3p_4^2 - p_1p_3p_4^2 - p_1p_2p_4^2 - p_1p_3p_4^3 - p_1p_2p_4^3 - p_2p_3p_4^3 - p_1p_2p_3-$$
$$- p_3p_4 - p_2p_3 - p_1p_4 - p_2p_4 - p_1p_4^2 - p_3p_4^2 - p_1p_4^3 - p_2p_4^2 - p_1p_3 - p_2p_4^3 - p_3p_4^3 - p_1p_2-$$
$$- p_4 - p_3 - p_4^2 - p_2 - p_4^3 - p_1 - p_2p_3p_4.$$

Remark 4.5. *The converse of the Part (14) of Theorem 4.3 is not true. For example if $n = p_1p_2p_3p_4^4$ the conditions of this assertion are not satisfied but there is a Hamiltonian cycle in $\mathfrak{ag}(\mathcal{L})$.*

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Saeid Bagheri

Department of Mathematics, Faculty of Mathematical Sciences, Malayer University, P.O.Box: 65719-95863, Malayer, Iran.

Email: bagheri_saeid@yahoo.com, s.bagheri@malayeru.ac.ir

Mahtab Koohi Kerahroodi

Department of Mathematics, Faculty of Mathematical Sciences, Malayer University, P.O.Box: 65719-95863, Malayer, Iran.

Email: mhbkoohi@gmail.com

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