RECOGNITION OF THE SIMPLE GROUPS $\text{PSL}_2(q)$ BY CHARACTER DEGREE GRAPH AND ORDER

Z. AKHLAGHI AND M. KHATAMI* AND B. KHOSRAVI

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Abstract. Let $G$ be a finite group, and $\text{Irr}(G)$ be the set of complex irreducible characters of $G$. Let $\rho(G)$ be the set of prime divisors of character degrees of $G$. The character degree graph of $G$, which is denoted by $\Delta(G)$, is a simple graph with vertex set $\rho(G)$, and we join two vertices $r$ and $s$ by an edge if there exists a character degree of $G$ divisible by $rs$. In this paper, we prove that if $G$ is a finite group such that $\Delta(G) = \Delta(\text{PSL}_2(q))$ and $|G| = |\text{PSL}_2(q)|$, then $G \cong \text{PSL}_2(q)$.

1. Introduction

Let $G$ be a finite group, and $\text{Irr}(G)$ be the set of complex irreducible characters of $G$. The set of character degrees of $G$ is denoted by $\text{cd}(G)$, and the set of prime divisors of elements of $\text{cd}(G)$ is denoted by $\rho(G)$. It is well-known that some information about the structure of the group $G$ can be obtained from $\text{cd}(G)$. A useful way to study the set of character degrees of a group $G$, is attaching graphs to $\text{cd}(G)$. One of these graphs that has been studied by different authors, is the character degree graph that was first defined in [9]. The character degree graph of the group $G$, which is denoted by $\Delta(G)$, is a graph with vertex set $\rho(G)$, and two distinct vertices $p$ and $q$ are adjacent if and only if there exists $\chi \in \text{Irr}(G)$ such that $pq$ divides $\chi(1)$.

In [4], it has been proved that the simple group $\text{PSL}_2(p)$ where $p$ is a prime, is uniquely determined by its order and its largest and second largest irreducible character degrees. As a consequence of this result, the simple group $\text{PSL}_2(p)$ is uniquely determined by its character degree graph and its order. Then, in [5] the recognizability of the simple groups of order less that 6000, by order and character degree graphs has been proved. Also in [6], the authors showed that the simple groups $\text{PSL}_2(p^2)$ for

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*Corresponding author.
odd prime $p$, are uniquely determined by their order and their character degree graphs. In this paper, we continue this investigation for finite simple groups $\text{PSL}_2(q)$, where $q$ is a prime power:

**Main Theorem.** Let $G$ be a group such that $\Delta(G) = \Delta(\text{PSL}_2(q))$ and $|G| = |\text{PSL}_2(q)|$, where $q \geq 4$ is a prime power. Then $G \cong \text{PSL}_2(q)$.

All characters in this paper are complex characters and all graphs are finite and simple. For an integer $n$, we write $\pi(n)$ for the set of all prime divisors of $n$. We denote by $\pi(G)$, the set of all prime divisors of $|G|$. For every integer $n$ and every set of primes $\pi$, the $\pi$-part of $n$ is denoted by $n_{\pi}$. If $N$ is a normal subgroup of $G$, then the inertia group of $\theta \in \text{Irr}(N)$ in $G$ is denoted by $I_G(\theta)$ and $\text{Irr}(G|\theta)$ is the set of all irreducible constituent characters of $\theta^G$ and by $\text{cd}(G|\theta)$ we mean, the set of degrees of characters in $\text{Irr}(G|\theta)$.

2. Preliminary Results

**Lemma 2.1.** ([10, Theorem 2.7]) Let $p$ and $q$ be two different primes and put $\pi = \{p, q\}$. Let $S$ be a finite simple non-abelian group and assume that $S \subseteq G \leq \text{Aut}(S)$, where $|G/S| = p$, $p$ does not divide $|S|$ and $q$ divides $|S|$. Assume that $pq$ does not divide $\chi(1)$ for every $\chi \in \text{Irr}(G)$. Then $S$ is a finite simple group of Lie type in characteristic $q$, and $G$ does not have any abelian subgroup $H$ with $|H_\pi| = |G_\pi|$.

**Lemma 2.2.** Assume $N$ is a normal subgroup of a finite group $G$ and assume that $G/N \cong K/N \times H/N$ such that all of the Sylow subgroups of $H/N$ are cyclic and $(|K/N||H/N|) = 1$. If $\theta \in \text{Irr}(N)$ is $G$-invariant, then either every element in $\text{cd}(G|\theta)$ is divisible by some $x \in \pi(K/N)$; or $\lambda(1)\theta(1) \in \text{cd}(G|\theta)$, for every $\lambda \in \text{Irr}(G/N)$.

**Proof.** First suppose that $\theta$ does not extend to $K$, therefore for every $\chi \in \text{Irr}(K|\theta)$ we have $\chi_N = e\theta$, where $e \neq 1$ is a divisor of $|K : N|$. Hence for every element $a \in \text{cd}(K|\theta)$ there exists a prime $x \in \pi(K/N)$ that divides $a$. So we may assume $\theta$ extends to $K$. Since $(|K/N||H/N|) = 1$ and all of Sylow subgroups of $H/N$ are cyclic, then by [2, p. 295, Theorem 22.3] for every Sylow subgroup $P/N$ of $G/N$, $\theta$ extends to $P$. Then by [3, Corollary 11.31] $\theta$ extends to $G$. Therefore by Gallagher’s theorem [3, Corollary 6.17], $\lambda(1)\theta(1) \in \text{cd}(G|\theta)$, for every $\lambda \in \text{Irr}(G/N)$.

**Lemma 2.3.** (Zsigmondy Theorem [15]) Let $p$ be a prime and let $n$ be a positive integer. Then one of the following holds:

(i) there is a primitive prime $p'$ for $p^n - 1$, that is, $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$,

(ii) $p = 2, n = 1$ or 6,

(iii) $p$ is a Mersenne prime and $n = 2$.

The following results are well-known and we will make use of them without giving more reference.

By Itô-Michler theorem, we know that a group $G$ has a normal abelian Sylow $p$-subgroup if and only if $p \not\in \rho(G)$ (see [2]). By Pálfy’s Condition, if $G$ is a solvable group and $\pi \subseteq \rho(G)$ such that $|\pi| \geq 3$, then there exist primes $p, q \in \pi$ and a degree $a \in \text{cd}(G)$ such that $pq$ divides $a$ ([7, Theorem 4.1]).
3. Proof of the main theorem

Note that the groups PSL$_2(2)$ and PSL$_2(3)$ are not simple, so we consider the groups PSL$_2(q)$ for $q \geq 4$.

**Theorem 3.1.** Let $G$ be a finite group such that $\Delta(G) = \Delta(PSL_2(q))$ and $|G| = |PSL_2(q)|$, where $q = p^f \geq 4$ is a prime power. Then $G \cong PSL_2(q)$.

**Proof.** If $f = 1$ or 2, then the theorem is true by the main results of [4, 6]. So we may assume that $f \geq 3$.

First assume $G$ is solvable. If $\pi(q - 1) \setminus \pi(q + 1) \neq \emptyset$ and $\pi(q + 1) \setminus \pi(q - 1) \neq \emptyset$, then using Pálly’s Condition we get a contradiction. Therefore we may assume there exists $\epsilon \in \{\pm 1\}$ such that $\pi(q + \epsilon) \subseteq \pi(q - \epsilon)$. Therefore, either $q = 9$; or $q = p$ is a Mersenne prime or a Fermat prime, which is a contradiction since $f \geq 3$. So from now on we assume $G$ is nonsolvable.

Let $N$ be the radical solvable subgroup of $G$, and $M/N$ be a chief factor of $G$. So $M/N \cong S^m$, is the direct product of $m$ copies of a nonabelian simple group $S$. Set $C/N = C_{G/N}(M/N)$. Then $C \leq G$, $MC/C \cong M/N$ and $MC/C$ is the unique minimal normal subgroup of $G/C$.

We claim that $p \in \pi(M/N)$. On the contrary, assume $p \notin \pi(M/N)$. First suppose that $p \in \pi(C/N)$. By Itô-Michler theorem we have $\pi(C/N) = \rho(C/N)$, so by the fact that $C/N \times M/N \leq G/N$ we have $p$ would be adjacent to all of the primes in $\pi(M/N)$, which is not possible, as $p$ is an isolated vertex of $\Delta(G)$.

Let $p \in \pi(G/C)$. Note that $S^m \cong MC/C \leq G/C \hookrightarrow Aut(S^m)$. If $m > 1$, then by the main theorem of [8] we have $\Delta(G/C)$ is complete. Since $\rho(G/C) = \pi(G/C)$ by Itô-Michler theorem, it follows that $p$ is adjacent to all primes in $\rho(M/N)$, which is impossible. Hence we may assume $m = 1$. Let $T/C$ be a subgroup of $G/C$ such that $|T/C : MC/C| = p$. By Lemma 2.1, there is at most one vertex in $\rho(S)$ which is not adjacent to $p$, which is a contradiction by the fact that $p$ is an isolated vertex of $\Delta(G)$.

Therefore $p \in \rho(N)$. Let $\theta \in Irr(N)$ such that $p \mid \theta(1)$ and let $T \subseteq M$, such that $T/N \cong S$. Using [12, Lemma 4.2] we have either $\chi(1)/\theta(1)$ is divisible by two distinct primes in $\pi(T/N)$ for some $\chi \in Irr(T|\theta)$, or $\theta$ is extendible to $\theta_0 \in Irr(T)$ and $T/N \cong A_5$ or PSL$_2(8)$. In both cases, $p$ is adjacent to some other prime in $\rho(S)$, a contradiction. Hence $p \in \pi(M/N)$, as we claimed.

Then $C = N$, since otherwise by the fact $C/N \times M/N \leq G/N$ we see that every primes in $\pi(M/N)$ would be adjacent to all primes in $\pi(C/N)$, which is impossible as $p$ is isolated.

Note that $M/N$ is a direct product of $m$ copies of nonabelian simple group $S$. If $m > 1$, then $p$ is adjacent to other vertices in $\rho(S) = \pi(S)$, which is a contradiction. So $m = 1$, and $M/N \cong S$. Since $\Delta(S)$ is disconnected, by [14, Theorem 6.1] we have $M/N \cong S \cong PSL_2(r^k)$, for some prime $r$ and some integer $k$. Now since $p$ is an isolated vertex of $\Delta(S)$, by considering the connected components of the character degree graph of PSL$_2(r^k)$ in [14, Theorem 5.2] we get that $r = p$; or $p$ is odd, $r = 2$ and $\pi(2^k + \epsilon) = \{p\}$, for $\epsilon = \pm 1$.

First suppose that $M/N \cong PSL_2(p^k)$, where $k$ is an integer. So $|S| = p^k(p^{2k} - 1)/(2, p^k - 1)$ and $\pi(p^{2k} - 1) \subseteq \pi(p^{2f} - 1)$. If $p^{2k} - 1$ has a primitive prime divisor, then it is easy to see that $k$ divides $f$.
If $p^{2k} - 1$ does not have a primitive prime divisor, then using Lemma 2.3, either $k = 1$ or $(k, p) = (3, 2)$, in both cases we get $k | f$. We claim that $k = f$. Arguing by contradiction, suppose that $k < f$.

Assume $t \in \rho(G) \setminus \rho(S)$. In the following we prove that $t$ is adjacent to all primes in $\pi(p^{2k} - 1)$. Let $t \in \pi(G/N) = \rho(G/N)$, then by the fact that $t \nmid |S|$, it follows that $t$ is a divisor of $|\text{Out}(\text{PSL}(2^k))|$. Therefore, by [13, Theorem A], $t$ is adjacent to every divisor of $p^{2k} - 1$. So we may assume $t \in \rho(G) \setminus \rho(G/N) \subseteq \rho(N)$. Let $\theta$ be an irreducible character of $N$ such that $t$ divides $\theta(1)$. Assume $\theta$ is not $M$-invariant. So $I = I_M(\theta) < M$. We know that every element of $\text{cd}(M|\theta)$ is divided by $|M : I|\theta(1)$, by Clifford’s corresponding theorem. Since $I/N$ is a proper subgroup of $M/N \cong \text{PSL}(2^k)$, there exists a maximal subgroup $T/N$ of $M/N$ such that $I/N \leq T/N \leq M/N$. So $|M : T|\theta(1)$ is a divisor of all of the elements in $\text{cd}(M|\theta)$. By [1, Hauptsatz II.8.27], the maximal subgroups of $\text{PSL}(2^k)$ are:

$$C_2^k \times C_{2^k-1}, D_{2(2^k-1)}, D_{2(2^k+1)}, \text{PGL}(2^k),$$

where $k/b = n \geq 2$ is a prime, and the maximal subgroups of $\text{PSL}(p^k)$, where $p$ is an odd prime, are $C_p \times C_{(p^k-1)/2}$, $D_{p^k+1}$ for $p^k \geq 13$, $D_{p^k+1}$ for $p^k \neq 7, 9$, $\text{PGL}(p^k)$ where $k/b = 2$, $\text{PSL}(p^a)$ where $k/a = n > 2$ is a prime, $A_5$ for $p^k \equiv \pm 1 \pmod{10}$, where either $k = 1$ or $k = 2$ and $p \equiv \pm 3 \pmod{10}$, $A_4$ for $p^k = p \equiv \pm 3 \pmod{8}$ and $p^k \neq \pm 1 \pmod{10}$, $S_4$ for $p^k = p \equiv \pm 1 \pmod{8}$.

Note that if $|M : T|$ is divided by $p$, then $p$ is adjacent to $t$ which is not possible. So the only possibility is $|M : T| = p^k + 1$. Therefore, $t$ is adjacent to all primes in $\pi(p^k + 1)$. Note that in this case $T/N$ is a Frobenius group with Frobenius kernel of order $p^k$ and a cyclic Frobenius complement of order $(p^k - 1)/(2, p - 1)$. Since $p \mid |M : I|$, it is easy to see that either $|I/N| = p^k$, or $I/N \cong K/N \rtimes H/N$, where $K/N$ is of order $p^k$, $(|K/N|, |H/N|) = 1$ and all of Sylow subgroups of $H/N$ are cyclic. If $|I/N| = p^k$, then $|M : I| = (p^{2k} - 1)/(2, p - 1)$, and by the fact that every element of $\text{cd}(M|\theta)$ is divided by $|M : I|\theta(1)$, we get that $t$ is adjacent to all primes in $\pi(p^{2k} - 1)$ as required. So assume that $|I/N| \cong K/N \rtimes H/N$, where $K/N$ is of order $p^k$, $(|K/N|, |H/N|) = 1$ and all of Sylow subgroups of $H/N$ are cyclic. Now using Lemma 2.2, we have $t$ is adjacent either to $p$ or to all primes in $\pi((p^k - 1)/s)$ where $s = |T : I| \geq 1$. Since $p$ is an isolated vertex of $\Delta(G)$, we get the first case is not possible and so $t$ is adjacent to all primes in $\pi((p^k - 1)/s)$ where $s = |T : I|$. On the other hand, we have $t$ is adjacent to all prime divisors of $|M : I| = s(p^k + 1)$, so $t$ is adjacent to all primes in $\pi(p^{2k} - 1)$. Hence we get our desired result.

So we may assume $\theta$ is $M$-invariant. If $\theta$ is extendible to $M$, then using Gallagher’s theorem we get $t$ is adjacent to $p$, a contradiction. So $\theta$ is not extendible to $M$. If $p = 2$, then since $\text{PSL}(2, 2^k)$ has trivial Schur multiplier, it follows from [3, Theorem 11.7] that $\theta$ is extendible to $M$, which is a contradiction. So $p$ is odd. If $p^k \neq 9$, then the Schur cover of $\text{PSL}(2^k)$ is $\text{SL}(2^k)$. By the theory of character triple isomorphisms in [3, Chapter 11], we deduce that $G$ has an irreducible character whose degree is divisible by $t(p^k \pm 1)$, which is our desired result. Now assume that $p^k = 9$. Then $(M, N, \theta)$ is character triple isomorphic to the triple $(L, A, \lambda)$ by [3, Chapter 11], where $L$ and $A$ are Schur cover and Schur multiplier of $\text{PSL}(9)$, respectively, and $\lambda \in \text{Irr}(A)$ is nontrivial. Then for any $\chi \in \text{Irr}(L|\lambda)$, we have $\theta(1)\chi(1)/\lambda(1) = \theta(1)\chi(1) \in \text{cd}(M|\theta)$. Since 3 is an isolated vertex of $\Delta(G)$, we
deduce that $3 \nmid \chi(1)$, for every $\chi \in \operatorname{Irr}(L|\lambda)$. So, it is easy to get by GAP that $\chi(1) \in \{4, 5, 8, 10\}$. Therefore, $|\operatorname{PSL}_2(9)| = |L : A| = \lambda^L(1) = \sum_{i=1}^d f_i \chi_i(1)$, for some integers $f_i$. By an easy computation, there exists $\chi_i, \chi_j \in \operatorname{Irr}(L|\lambda)$ such that $2 \mid \chi_i(1)$ and $5 \mid \chi_j(1)$. So by the above argument $t$ is adjacent to all primes in $\pi(9^2 - 1) = \{2, 5\}$, as required.

Now assume $p^f - 1$ and $p^{2f} - 1$ have primitive prime divisors and we denote those numbers by $x$ and $y$, respectively. By above discussion we have both $x$ and $y$ are adjacent to all primes in $\pi(p^{2k} - 1)$. Hence $\pi(p^{2k} - 1) = \{2\}$, which implies that $k = 1$ and $p = 3$. So $M/N$ is solvable which is not possible.

So we may assume that either $p^f - 1$ or $p^{2f} - 1$ does not have a primitive prime divisor. Since $f \geq 3$, we have either $p = 2$ and $f = 3$; or $p = 2$ and $f = 6$.

If the first case occurs, then $k = 1$, which implies that $M/N$ is a solvable group, a contradiction. If the last case occurs, then either $k = 2$ or $k = 3$ and $p^{2f} - 1$ has a primitive prime divisor that we call it $y$. Then $y$ would be adjacent to all primes $2^{2k} - 1$ and so $y$ is adjacent to $3 \in \pi(p^f - 1)$, a contradiction. Therefore $f = k$ and our claim is proved.

So $S \cong \operatorname{PSL}_2(q)$. Now by the hypothesis $|G| = |\operatorname{PSL}_2(q)|$, we get that $N = 1$ and $G = M \cong \operatorname{PSL}_2(q)$, as required.

Now suppose that $p$ is odd, $M/N \cong \operatorname{PSL}_2(2^k)$ and $\pi(2^k + \epsilon) = \{p\}$, for an integer $k$ and $\epsilon = \pm 1$. Therefore by looking at the character degree graph of $\operatorname{PSL}_2(q)$ we have $\pi(2^{(2k - \epsilon)}) \subseteq \pi(p^f + \nu)$, for $\nu = \pm 1$. First assume that there exists $t \not\mid \pi(p^f - \nu)$. So $t \not\mid \rho(G) \setminus \rho(S)$. If $t \mid \rho(G/N) = \rho(G/N)$, then since $t \mid |M/N|$ we have $t$ is a divisor of $|\operatorname{Out}(\operatorname{PSL}_2(2^k))|$. So by [13, Theorem A], one can get that $t$ is adjacent to every prime divisors of $2^{2k} - 1$, which contradicts the fact that $\pi(2^k + \epsilon) = \{p\}$ and $\pi(2^k - \epsilon) \subseteq \pi(p^f + \nu)$. So $t \mid \rho(G) \setminus \rho(G/N) \subseteq \rho(N)$. Let $\theta$ be an irreducible character of $N$ such that $t \mid \theta(1)$. Assume $\theta$ is $M$-invariant. Since also $\operatorname{PSL}_2(2^k)$ has trivial Schur multiplier, it follows from [3, Theorem 11.7] that $\theta$ is extendible to $M$. Then by Gallagher’s theorem we get that $t$ is adjacent to $p$, which is impossible. So we may assume $\theta$ is not $M$-invariant. Let $I = I_M(\theta) < M$. By Clifford’s corresponding theorem, every element of $\operatorname{cd}(M|\theta)$ is divisible by $|I : I|\theta(1)$. Suppose that $T/N$ is a maximal subgroup of $M/N$ such that $I/N \leq T/N \leq M/N$. So every element of $\operatorname{cd}(M|\theta)$ is divisible by $|M : T|\theta(1)$. By considering the maximal subgroups of $\operatorname{PSL}_2(2^k)$, it follows that $t$ is adjacent either to $p$; or to an odd prime divisor of $2^k - \epsilon$, a contradiction.

Therefore $\pi(p^f - \nu) = \{2\}$, and consequently $p^f - 1$ or $p^{2f} - 1$ do not have primitive prime divisors. Since $p$ is odd, Lemma 2.3 implies that $f = 1, 2$, which contradicts our assumption $f \geq 3$. Hence this case is impossible and the theorem is proved.

\[\square\]

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Z. Akhlaghi
Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 15914, Tehran, Iran
Email: z_akhlaghi@aut.ac.ir

M. Khatami
Department of Mathematics, University of Isfahan, Isfahan, 81746-73441, Isfahan, Iran
Email: m.khatami@sci.ui.ac.ir

B. Khosravi
Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 15914, Tehran, Iran
Email: khosravibbb@yahoo.com