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RECOGNITION OF THE SIMPLE GROUPS $\mathrm{PSL}_2(q)$ BY CHARACTER DEGREE GRAPH AND ORDER

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Abstract. Let G be a finite group, and $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G. Let $\rho(G)$ be the set of prime divisors of character degrees of G. The character degree graph of G, which is denoted by $\Delta(G)$, is a simple graph with vertex set $\rho(G)$, and we join two vertices r and s by an edge if there exists a character degree of G divisible by rs. In this paper, we prove that if G is a finite group such that $\Delta(G) = \Delta(\mathrm{PSL}_2(q))$ and $|G| = |\mathrm{PSL}_2(q)|$, then $G \cong \mathrm{PSL}_2(q)$.

1. Introduction

Let G be a finite group, and $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G. The set of character degrees of G is denoted by $\mathrm{cd}(G)$, and the set of prime divisors of elements of $\mathrm{cd}(G)$ is denoted by $\rho(G)$. It is well-known that some information about the structure of the group G can be obtained from $\mathrm{cd}(G)$. A useful way to study the set of character degrees of a group G, is attaching graphs to $\mathrm{cd}(G)$. One of these graphs that has been studied by different authors, is the character degree graph that was first defined in [9]. The character degree graph of the group G, which is denoted by $\Delta(G)$, is a graph with vertex set $\rho(G)$, and two distinct vertices p and q are adjacent if and only if there exists $\chi \in \mathrm{Irr}(G)$ such that pq divides $\chi(1)$.

In [4], it has been proved that the simple group $\mathrm{PSL}_2(p)$ where p is a prime, is uniquely determined by its order and its largest and second largest irreducible character degrees. As a consequence of this result, the simple group $\mathrm{PSL}_2(p)$ is uniquely determined by its character degree graph and its order. Then, in [5] the recognizability of the simple groups of order less that 6000, by order and character degree graphs has been proved. Also in [6], the authors showed that the simple groups $\mathrm{PSL}_2(p^2)$ for

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odd prime p, are uniquely determined by their order and their character degree graphs. In this paper, we continue this investigation for finite simple groups $\mathrm{PSL}_2(q)$, where q is a prime power:

Main Theorem. Let G be a group such that $\Delta(G) = \Delta(\mathrm{PSL}_2(q))$ and $|G| = |\mathrm{PSL}_2(q)|$, where $q \geq 4$ is a prime power. Then $G \cong \mathrm{PSL}_2(q)$.

All characters in this paper are complex characters and all graphs are finite and simple. For an integer n, we write $\pi(n)$ for the set of all prime divisors of n. We denote by $\pi(G)$, the set of all prime divisors of $|G|$. For every integer n and every set of primes π, the π-part of n is denoted by n_π. If N is a normal subgroup of G, then the inertia group of $\theta \in \mathrm{Irr}(N)$ in G is denoted by $I_G(\theta)$ and $\mathrm{Irr}(G|\theta)$ is the set of all irreducible constituent characters of θ^G and by $\mathrm{cd}(G|\theta)$ we mean, the set of degrees of characters in $\mathrm{Irr}(G|\theta)$.

2. Preliminary Results

Lemma 2.1. ([10, Theorem 2.7]) *Let p and q be two different primes and put $\pi = \{p, q\}$. Let S be a finite simple non-abelian group and assume that $S \trianglelefteq G \leq \mathrm{Aut}(S)$, where $|G/S| = p$, p does not divide $|S|$ and q divides $|S|$. Assume that pq does not divide $\chi(1)$ for every $\chi \in \mathrm{Irr}(G)$. Then S is a finite simple group of Lie type in characteristic q, and G does not have any abelian subgroup H with $|H|_\pi = |G|_\pi$.*

Lemma 2.2. *Assume N is a normal subgroup of a finite group G and assume that $G/N \cong K/N \rtimes H/N$ such that all of the Sylow subgroups of H/N are cyclic and $(|K/N|, |H/N|) = 1$. If $\theta \in \mathrm{Irr}(N)$ is G-invariant, then either every element in $\mathrm{cd}(\mathrm{G}|\theta)$ is divisible by some $x \in \pi(K/N)$; or $\lambda(1)\theta(1) \in \mathrm{cd}(\mathrm{G}|\theta)$, for every $\lambda \in \mathrm{Irr}(G/N)$.*

Proof. First suppose that θ does not extend to K, therefore for every $\chi \in \mathrm{Irr}(K|\theta)$ we have $\chi_N = e\theta$, where $e \neq 1$ is a divisor of $|K : N|$. Hence for every element $a \in \mathrm{cd}(\mathrm{K}|\theta)$ there exists a prime $x \in \pi(K/N)$ that divides a. So we may assume θ extends to K. Since $(|K/N|, |H/N|) = 1$ and all of Sylow subgroups of H/N are cyclic, then by [2, p. 295, Theorem 22.3] for every Sylow subgroup P/N of G/N, θ extends to P. Then by [3, Corollary 11.31] θ extends to G. Therefore by Gallagher's theorem [3, Corollary 6.17], $\lambda(1)\theta(1) \in \mathrm{cd}(\mathrm{G}|\theta)$, for every $\lambda \in \mathrm{Irr}(G/N)$. □

Lemma 2.3. (Zsigmondy Theorem [15]) *Let p be a prime and let n be a positive integer. Then one of the following holds:*

(i) there is a primitive prime p' for $p^n - 1$, that is , $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$,

(ii) $p = 2$, $n = 1$ or 6,

(iii) p is a Mersenne prime and $n = 2$.

The following results are well-known and we will make use of them without giving more reference.

By Itô-Michler theorem, we know that a group G has a normal abelian Sylow p-subgroup if and only if $p \notin \rho(G)$ (see [2]). By Pálfy's Condition, if G is a solvable group and $\pi \subseteq \rho(G)$ such that $|\pi| \geq 3$, then there exist primes $p, q \in \pi$ and a degree $a \in \mathrm{cd}(G)$ such that pq divides a ([7, Theorem 4.1]).

3. Proof of the main theorem

Note that the groups $\mathrm{PSL}_2(2)$ and $\mathrm{PSL}_2(3)$ are not simple, so we consider the groups $\mathrm{PSL}_2(q)$ for $q \geq 4$.

Theorem 3.1. *Let G be a finite group such that $\Delta(G) = \Delta(\mathrm{PSL}_2(q))$ and $|G| = |\mathrm{PSL}_2(q)|$, where $q = p^f \geq 4$ is a prime power. Then $G \cong \mathrm{PSL}_2(q)$.*

Proof. If $f = 1$ or 2, then the theorem is true by the main results of [4, 6]. So we may assume that $f \geq 3$.

First assume G is solvable. If $\pi(q-1) \setminus \pi(q+1) \neq \emptyset$ and $\pi(q+1) \setminus \pi(q-1) \neq \emptyset$, then using Pálfy's Condition we get a contradiction. Therefore we may assume there exists $\epsilon \in \{\pm 1\}$ such that $\pi(q+\epsilon) \subseteq \pi(q-\epsilon)$. Therefore, either $q = 9$; or $q = p$ is a Mersenne prime or a Fermat prime, which is a contradiction since $f \geq 3$. So from now on we assume G is nonsolvable.

Let N be the radical solvable subgroup of G, and M/N be a chief factor of G. So $M/N \cong S^m$, is the direct product of m copies of a nonabelian simple group S. Set $C/N = C_{G/N}(M/N)$. Then $C \trianglelefteq G$, $MC/C \cong M/N$ and MC/C is the unique minimal normal subgroup of G/C.

We claim that $p \in \pi(M/N)$. On the contrary, assume $p \notin \pi(M/N)$. First suppose that $p \in \pi(C/N)$. By Itȯ-Michler theorem we have $\pi(C/N) = \rho(C/N)$, so by the fact that $C/N \times M/N \trianglelefteq G/N$ we have p would be adjacent to all of the primes in $\pi(M/N)$, which is not possible, as p is an isolated vertex of $\Delta(G)$.

Let $p \in \pi(G/C)$. Note that $S^m \cong MC/C \leq G/C \hookrightarrow \mathrm{Aut}(S^m)$. If $m > 1$, then by the main theorem of [8] we have $\Delta(G/C)$ is complete. Since $\rho(G/C) = \pi(G/C)$ by Itȯ-Michler theorem, it follows that p is adjacent to all primes in $\rho(M/N)$, which is impossible. Hence we may assume $m = 1$. Let T/C be a subgroup of G/C such that $|T/C : MC/C| = p$. By Lemma 2.1, there is at most one vertex in $\rho(S)$ which is not adjacent to p, which is a contradiction by the fact that p is an isolated vertex of $\Delta(G)$.

Therefore $p \in \rho(N)$. Let $\theta \in \mathrm{Irr}(N)$ such that $p \mid \theta(1)$ and let $T \subseteq M$, such that $T/N \cong S$. Using [12, Lemma 4.2] we have either $\chi(1)/\theta(1)$ is divisible by two distinct primes in $\pi(T/N)$ for some $\chi \in \mathrm{Irr}(T|\theta)$, or θ is extendible to $\theta_0 \in \mathrm{Irr}(T)$ and $T/N \cong A_5$ or $\mathrm{PSL}_2(8)$. In both cases, p is adjacent to some other prime in $\rho(S)$, a contradiction. Hence $p \in \pi(M/N)$, as we claimed.

Then $C = N$, since otherwise by the fact $C/N \times M/N \trianglelefteq G/N$ we see that every primes in $\pi(M/N)$ would be adjacent to all primes in $\pi(C/N)$, which is impossible as p is isolated.

Note that M/N is a direct product of m copies of nonabelian simple group S. If $m > 1$, then p is adjacent to other vertices in $\rho(S) = \pi(S)$, which is a contradiction. So $m = 1$, and $M/N \cong S$. Since $\Delta(S)$ is disconnected, by [14, Theorem 6.1] we have $M/N \cong S \cong \mathrm{PSL}_2(r^k)$, for some prime r and some integer k. Now since p is an isolated vertex of $\Delta(S)$, by considering the connected components of the character degree graph of $\mathrm{PSL}_2(r^k)$ in [14, Theorem 5.2] we get that $r = p$; or p is odd, $r = 2$ and $\pi(2^k + \epsilon) = \{p\}$, for $\epsilon = \pm 1$.

First suppose that $M/N \cong \mathrm{PSL}_2(p^k)$, where k is an integer. So $|S| = p^k(p^{2k}-1)/(2, p^k - 1)$ and $\pi(p^{2k}-1) \subseteq \pi(p^{2f}-1)$. If $p^{2k}-1$ has a primitive prime divisor, then it is easy to see that k divides f.

If $p^{2k}-1$ does not have a primitive prime divisor, then using Lemma 2.3, either $k=1$ or $(k,p)=(3,2)$, in both cases we get $k \mid f$. We claim that $k=f$. Arguing by contradiction, suppose that $k<f$.

Assume $t \in \rho(G) \setminus \rho(S)$. In the following we prove that t is adjacent to all primes in $\pi(p^{2k}-1)$. Let $t \in \pi(G/N)=\rho(G/N)$, then by the fact that $t \nmid |S|$, it follows that t is a divisor of $|\mathrm{Out}(\mathrm{PSL}_2(p^k))|$. Therefore, by [13, Theorem A], t is adjacent to every divisor of $p^{2k}-1$. So we may assume $t \in \rho(G) \setminus \rho(G/N) \subseteq \rho(N)$. Let θ be an irreducible character of N such that t divides $\theta(1)$. Assume θ is not M-invariant. So $I=I_M(\theta)<M$. We know that every element of $\mathrm{cd}(M|\theta)$ is divided by $|M:I|\theta(1)$, by Clifford's corresponding theorem. Since I/N is a proper subgroup of $M/N \cong \mathrm{PSL}_2(p^k)$, there exists a maximal subgroup T/N of M/N such that $I/N \leq T/N \lneqq M/N$. So $|M:T|\theta(1)$ is a divisor of all of the elements in $\mathrm{cd}(M|\theta)$. By [1, Hauptsatz II.8.27], the maximal subgroups of $\mathrm{PSL}_2(2^k)$ are:

$$C_2^k \rtimes C_{2^k-1}, D_{2(2^k-1)}, D_{2(2^k+1)}, \mathrm{PGL}_2(2^b),$$

where $k/b=n\geq 2$ is a prime, and the maximal subgroups of $\mathrm{PSL}_2(p^k)$, where p is an odd prime, are $C_p^k \rtimes C_{(p^k-1)/2}$, D_{p^k-1} for $p^k \geq 13$, D_{p^k+1} for $p^k \neq 7,9$, $\mathrm{PGL}_2(p^b)$ where $k/b=2$, $\mathrm{PSL}_2(p^a)$ where $k/a=n>2$ is a prime, A_5 for $p^k \equiv \pm 1 \pmod{10}$, where either $k=1$ or $k=2$ and $p \equiv \pm 3 \pmod{10}$, A_4 for $p^k=p\equiv \pm 3 \pmod 8$ and $p^k \not\equiv \pm 1 \pmod{10}$, S_4 for $p^k=p\equiv \pm 1 \pmod 8$.

Note that if $|M:T|$ is divided by p, then p is adjacent to t which is not possible. So the only possibility is $|M:T|=p^k+1$. Therefore, t is adjacent to all primes in $\pi(p^k+1)$. Note that in this case T/N is a Frobenius group with Frobenius kernel of order p^k and a cyclic Frobenius complement of order $(p^k-1)/(2,p-1)$. Since $p \nmid |M:I|$, it is easy to see that either $|I/N|=p^k$, or $I/N \cong K/N \rtimes H/N$, where K/N is of order p^k, $(|K/N|,|H/N|)=1$ and all of Sylow subgroups of H/N are cyclic. If $|I/N|=p^k$, then $|M:I|=(p^{2k}-1)/(2,p-1)$, and by the fact that every element of $\mathrm{cd}(M|\theta)$ is divided by $|M:I|\theta(1)$, we get that t is adjacent to all primes in $\pi(p^{2k}-1)$ as required. So assume that $I/N \cong K/N \rtimes H/N$, where K/N is of order p^k, $(|K/N|,|H/N|)=1$ and all of Sylow subgroups of H/N are cyclic. Now using Lemma 2.2, we have t is adjacent either to p or to all primes in $\pi((p^k-1)/s)$ where $s=|T:I|\geq 1$. Since p is an isolated vertex of $\Delta(G)$, we get the first case is not possible and so t is adjacent to all primes in $\pi((p^k-1)/s)$ where $s=|T:I|$. On the other hand, we have t is adjacent to all prime divisors of $|M:I|=s(p^k+1)$, so t is adjacent to all primes in $\pi(p^{2k}-1)$. Hence we get our desired result.

So we may assume θ is M-invariant. If θ is extendible to M, then using Gallagher's theorem we get t is adjacent to p, a contradiction. So θ is not extendible to M. If $p=2$, then since $\mathrm{PSL}(2,2^k)$ has trivial Schur multiplier, it follows from [3, Theorem 11.7] that θ is extendible to M, which is a contradiction. So p is odd. If $p^k \neq 9$, then the Schur cover of $\mathrm{PSL}_2(p^k)$ is $\mathrm{SL}_2(p^k)$. By the theory of character triple isomorphisms in [3, Chapter 11], we deduce that G has an irreducible character whose degree is divisible by $t(p^k \pm 1)$, which is our desired result. Now assume that $p^k=9$. Then (M,N,θ) is character triple isomorphic to the triple (L,A,λ) by [3, Chapter 11], where L and A are Schur cover and Schur multiplier of $\mathrm{PSL}_2(9)$, respectively, and $\lambda \in \mathrm{Irr}(A)$ is nontrivial. Then for any $\chi \in \mathrm{Irr}(L|\lambda)$, we have $\theta(1)\chi(1)/\lambda(1)=\theta(1)\chi(1) \in \mathrm{cd}(M|\theta)$. Since 3 is an isolated vertex of $\Delta(G)$, we deduce that $3 \nmid \chi(1)$, for every $\chi \in \mathrm{Irr}(L|\lambda)$. So, it is easy to get by GAP that $\chi(1) \in \{4, 5, 8, 10\}$. Therefore, $|\mathrm{PSL}_2(9)| = |L : A| = \lambda^L(1) = \Sigma_{i=1}^4 f_i\chi_i(1)$, for some integers f_i. By an easy computation, there exists $\chi_i, \chi_j \in \mathrm{Irr}(L|\lambda)$ such that $2 \mid \chi_i(1)$ and $5 \mid \chi_j(1)$. So by the above argument t is adjacent to all primes in $\pi(9^2 - 1) = \{2, 5\}$, as required.

Now assume $p^f - 1$ and $p^{2f} - 1$ have primitive prime divisors and we denote those numbers by x and y, respectively. By above discussion we have both x and y are adjacent to all primes in $\pi(p^{2k} - 1)$. Hence $\pi(p^{2k} - 1) = \{2\}$, which implies that $k = 1$ and $p = 3$. So M/N is solvable which is not possible.

So we may assume that either $p^f - 1$ or $p^{2f} - 1$ does not have a primitive prime divisor. Since $f \geq 3$, we have either $p = 2$ and $f = 3$; or $p = 2$ and $f = 6$.

If the first case occurs, then $k = 1$, which implies that M/N is a solvable group, a contradiction. If the last case occurs, then either $k = 2$ or $k = 3$ and $p^{2f} - 1$ has a primitive prime divisor that we call it y. Then y would be adjacent to all primes $2^{2k} - 1$ and so y is adjacent to $3 \in \pi(p^f - 1)$, a contradiction. Therefore $f = k$ and our claim is proved.

So $S \cong \mathrm{PSL}_2(q)$. Now by the hypothesis $|G| = |\mathrm{PSL}_2(q)|$, we get that $N = 1$ and $G = M \cong \mathrm{PSL}_2(q)$, as required.

Now suppose that p is odd, $M/N \cong \mathrm{PSL}_2(2^k)$ and $\pi(2^k + \epsilon) = \{p\}$, for an integer k and $\epsilon = \pm 1$. Therefore by looking at the character degree graph of $\mathrm{PSL}_2(q)$ we have $\pi(2(2^k - \epsilon)) \subseteq \pi(p^f + \nu)$, for $\nu = \pm 1$. First assume that there exists $2 \neq t \in \pi(p^f - \nu)$. So $t \in \rho(G) \backslash \rho(S)$. If $t \in \pi(G/N) = \rho(G/N)$, then since $t \nmid |M/N|$ we have t is a divisor of $|\mathrm{Out}(\mathrm{PSL}_2(2^k))|$. So by [13, Theorem A], one can get that t is adjacent to every prime divisors of $2^{2k} - 1$, which contradicts the fact that $\pi(2^k + \epsilon) = \{p\}$ and $\pi(2^k - \epsilon) \subseteq \pi(p^f + \nu)$. So $t \in \rho(G) \setminus \rho(G/N) \subseteq \rho(N)$. Let θ be an irreducible character of N such that $t \mid \theta(1)$. Assume θ is M-invariant. Since also $\mathrm{PSL}_2(2^k)$ has trivial Schur multiplier, it follows from [3, Theorem 11.7] that θ is extendible to M. Then by Gallagher's theorem we get that t is adjacent to p, which is impossible. So we may assume θ is not M-invariant. Let $I = I_M(\theta) < M$. By Clifford's corresponding theorem, every element of $\mathrm{cd}(M|\theta)$ is divisible by $|M : I|\theta(1)$. Suppose that T/N is a maximal subgroup of M/N such that $I/N \leq T/N \lneqq M/N$. So every element of $\mathrm{cd}(M|\theta)$ is divisible by $|M : T|\theta(1)$. By considering the maximal subgroups of $\mathrm{PSL}_2(2^k)$, it follows that t is adjacent either to p; or to an odd prime divisor of $2^k - \epsilon$, a contradiction.

Therefore $\pi(p^f - \nu) = \{2\}$, and consequently $p^f - 1$ or $p^{2f} - 1$ do not have primitive prime divisors. Since p is odd, Lemma 2.3 implies that $f = 1, 2$, which contradicts our assumption $f \geq 3$. Hence this case is impossible and the theorem is proved. □

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