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B_π-CHARACTERS AND QUOTIENTS

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Abstract. Let π be a set of primes, and let G be a finite π-separable group. We consider the Isaacs B_π-characters. We show that if N is a normal subgroup of G, then $\mathrm{B}_\pi(G/N) = \mathrm{Irr}(G/N) \cap \mathrm{B}_\pi(G)$.

1. Introduction

All groups in this paper are finite. Throughout this paper π will be a set of primes and G will be a π-separable group. In [3], Isaacs defined the subset $\mathrm{B}_\pi(G)$ of $\mathrm{Irr}(G)$. In our paper [6] about a variation on Landau's theorem, we needed a basic fact about the characters in $\mathrm{B}_\pi(G)$ and quotients of G that seems not to have been proved anywhere. Since proving this fact in [6] would have been a distraction to the main point of that paper, we decided to establish the proof of this fact separately.

Many of the basic ideas about the set $\mathrm{B}_\pi(G)$ were proved in [3], and in fact, all of the facts we need about $\mathrm{B}_\pi(G)$ can be found in [3]. The papers [4] and [5] both give very good expository accounts about $\mathrm{B}_\pi(G)$ characters. Much of this paper is to provide the terminology, concepts, and definitions needed to explicitly define the set $\mathrm{B}_\pi(G)$.

Suppose G is a group and N is a normal subgroup of G. There is a bijection between the sets $\{\chi \in \mathrm{Irr}(G) \mid N \le \ker(\chi)\}$ and $\mathrm{Irr}(G/N)$ (see Lemma 2.22 of [2]). If $\chi \in \mathrm{Irr}(G)$ and $N \le \ker(\chi)$, then we write $\hat{\chi}$ to denote the corresponding character in $\mathrm{Irr}(G/N)$. With this in mind, we set the following notation: $\hat{\mathrm{B}}_\pi(G) = \{\hat{\chi} \mid \chi \in \mathrm{B}_\pi(G), N \le \ker(\chi)\}$. Notice that if we identify $\mathrm{Irr}(G/N)$ and $\mathrm{B}_\pi(G/N)$ with the appropriate subsets of $\mathrm{Irr}(G)$, then $\hat{\mathrm{B}}_\pi(G) = \mathrm{Irr}(G/N) \cap \mathrm{B}_\pi(G)$.

The goal of this paper is to prove the following:

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Theorem 1.1. *Suppose π is a set of primes and G is a π-separable group. If N is a normal subgroup of G, then $\mathrm{B}_\pi(G/N) = \hat{\mathrm{B}}_\pi(G)$.*

2. Results

The first idea that we need for defining the set $\mathrm{B}_\pi(G)$ are π-special characters. Following Gajendragadkar in [1], we say that a character $\chi \in \mathrm{Irr}(G)$ where G is *π-special* if $\chi(1)$ is a π-number and for every subnormal subgroup S of G, the irreducible constituents of χ_S have determinantal order that is a π-number.

In Proposition 7.1 of [1], Gajendragadkar proved that if $\alpha, \beta \in \mathrm{Irr}(G)$ are characters so that α is π-special and β is π'-special, then $\alpha\beta$ is irreducible, and this factorization is unique. I.e., if $\alpha\beta = \alpha'\beta'$ where α' is π-special and β' is π'-special, then $\alpha = \alpha'$ and $\beta = \beta'$.

Using [3], we say that $\chi \in \mathrm{Irr}(G)$ is *π-factored* if there exists a π-special character α and π'-special character β so that $\chi = \alpha\beta$. We take time out of our definitions to prove the following lemma regarding the kernels of π-factored characters which is key to our argument.

Lemma 2.1. *Suppose π is a set of primes and G is a π-separable group. If $\chi \in \mathrm{Irr}(G)$ satisfies $\chi = \alpha\beta$ where α is π-special and β is π'-special, then $\ker(\chi) = \ker(\alpha) \cap \ker(\beta)$.*

Proof. It is obvious that $\ker(\alpha) \cap \ker(\beta) \leq \ker(\chi) = K$. We need to show that $K \leq \ker(\alpha) \cap \ker(\beta)$. We first claim that $K \leq Z(\alpha) \cap Z(\beta)$. Suppose $g \in K$. Then $\alpha(g)\beta(g) = \chi(g) = \chi(1) = \alpha(1)\beta(1)$. Hence, $\alpha(1)\beta(1) = |\alpha(g)\beta(g)| = |\alpha(g)||\beta(g)|$. By Lemma 2.15(c) of [2], we know that $|\alpha(g)| \leq \alpha(1)$ and $|\beta(g)| \leq \beta(1)$. The previous equality implies that these inequalities must be equalities, so $g \in Z(\alpha)$ and $g \in Z(\beta)$. This proves the claim.

By Lemma 2.27 (c) of [2], we see that $\alpha_K = \alpha(1)\mu$ and $\beta_K = \beta(1)\nu$ for linear characters μ and ν in $\mathrm{Irr}(K)$. Because α is π-special, μ must have π-order and because β is π'-special, ν must have π'-order. For $g \in K$, this implies that $\mu(g)$ is a π root of unity and $\nu(g)$ is a π'-root of unity. We have $\alpha(1)\beta(1) = \chi(1) = \chi(g) = \alpha(g)\beta(g) = \alpha(1)\mu(g)\beta(1)\nu(g)$. This implies that $\nu(g)\mu(g) = 1$. The only way that the product of a π-root of unity and a π'-root can equal 1 is if they are both 1. I.e., we must have $\mu(g) = \nu(g) = 1$. This implies that $\alpha(1) = \alpha(g)$ and $\beta(1) = \beta(g)$. Therefore, $g \in \ker(\alpha) \cap \ker(\beta)$ as desired. □

Returning to our definitions, we fix the character $\chi \in \mathrm{Irr}(G)$. We say that (S, σ) is a *subnormal pair* for χ if S is a subnormal subgroup of G and σ is an irreducible constituent of χ_S. In addition, we say that (S, σ) is *π-factored* if σ is π-factored. We can define a partial ordering on the subnormal pairs for χ by $(S, \sigma) \leq (T, \tau)$ if $S \leq T$ and σ is a constituent of τ_S.

Notice that $(1, 1_1)$ is a π-factored subnormal pair for χ, so there exists a maximal π-factored subnormal pair for χ with respect to the partial ordering. It is shown in Theorem 3.2 of [3] that the set of maximal π-factored subnormal pairs for χ are conjugate in G. Let (S, σ) be a maximal π-factored subnormal pair for χ, and let T be the stabilizer of (S, σ) in G. It is shown in Theorem 4.4 of [3] that there is a unique character $\tau \in \mathrm{Irr}(T \mid \sigma)$ so that $\tau^G = \chi$.

We can now define the π-*nucleus* for χ. If χ is π-factored, then (G,χ) is the nucleus for χ. If χ is not π-factored, then let (S,σ) be a maximal π-factored subnormal pair for χ. Let T be the stabilizer of (S,σ) in G, and let $\tau \in \mathrm{Irr}(T \mid \sigma)$ so that $\tau^G = \chi$. By Lemma 4.5 of [3], we know that $T < G$, so we can inductively define the π-nucleus of χ to be the π-nucleus of τ. Because the maximal π-factored subnormal pairs are all conjugate, it follows that the π-nucleus for χ is well-defined up to conjugacy. (See the argument on page 108 of [3].)

We are now ready to state the definition of $\mathrm{B}_\pi(G)$. We still have the character χ. We take (X,η) to be a π-nucleus for χ, and by definition η must be π-factored. The set $\mathrm{B}_\pi(G)$ is defined to be those characters $\chi \in \mathrm{Irr}(G)$ where (X,η) is a π-nucleus for χ and η is π-special. This is the statement of Definition 5.1 of [3].

We now give a lemma that connects a π-nucleus with the quotient.

Lemma 2.2. *Let π be a set of primes and let G be a π-separable group. Suppose that N is a normal subgroup of G. If $\chi \in \mathrm{Irr}(G)$ with $N \le \ker(\chi)$ has π-nucleus (X,η), then $(X/N,\hat{\eta})$ is a π-nucleus for $\hat{\chi} \in \mathrm{Irr}(G/N)$.*

Proof. If $(X,\eta) = (G,\chi)$, then this is obvious. Thus, we may assume that $X < G$. Let (S,σ) be a maximal π-factored subnormal pair for χ with stabilizer T and character $\tau \in \mathrm{Irr}(T \mid \sigma)$ so that $\tau^G = \chi$ and (X,η) is a π-nucleus for τ. Notice that $(N, 1_N)$ is a π-factored subnormal pair for χ, so it is contained in a maximal such pair. As we mentioned above, the maximal π-factored subnormal pairs for χ are all conjugate. Since N is normal, this implies that $N \le S$, and thus, $(N,1_N) \le (S,\sigma)$. Because σ is a constituent of χ_S, we see that $N \le \ker(\sigma)$. By Lemma 2.1, we see that $\hat{\sigma}$ is π-factored as a character in $\mathrm{Irr}(S/N)$. Notice that $(S/N,\hat{\sigma}) \le (S^*/N,\hat{\sigma}^*)$ if and only if $(S,\sigma) \le (S^*,\sigma^*)$, and by Lemma 2.1, $\hat{\sigma}^*$ is π-factored in $\mathrm{Irr}(S^*/N)$ if and only if σ is π-factored in $\mathrm{Irr}(S^*)$. Therefore, $(S/N,\hat{\sigma})$ must be a maximal π-factored subnormal pair for $\hat{\chi}$. It is immediate that T/N will be the stabilizer for $(S/N,\hat{\sigma})$ in G/N and that $\hat{\tau}$ is the unique character in $\mathrm{Irr}(T/N \mid \hat{\sigma})$ that induces $\hat{\chi}$. By induction, $(X/N,\hat{\eta})$ will the π-nucleus for $\hat{\tau}$, and thus, $(X/N,\hat{\eta})$ will be a π-nucleus for $\hat{\chi}$. ☐

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Note that $\mathrm{B}_\pi(G/N) \subseteq \mathrm{Irr}(G/N)$. Hence, it suffices to show for $\chi \in \mathrm{Irr}(G)$ with $N \le \ker(\chi)$ that $\chi \in \mathrm{B}_\pi(G)$ if and only if $\hat{\chi} \in \mathrm{B}_\pi(G/N)$. Suppose $\chi \in \mathrm{Irr}(G/N)$. Let (X,η) be a π-nucleus for χ. By Lemma 2.2, $(X/N,\hat{\eta})$ is a nucleus for $\hat{\chi}$. We know that $\chi \in \mathrm{B}_\pi(G)$ if and only if η is π-special and $\hat{\chi} \in \mathrm{B}_\pi(G/N)$ if and only if $\hat{\eta}$ is π-special. Using the definition of π-special, we see that η is π-special if and only if $\hat{\eta}$ is π-special, and this proves the theorem. ☐

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