B\(\pi\)-CHARACTERS AND QUOTIENTS

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Abstract. Let \(\pi\) be a set of primes, and let \(G\) be a finite \(\pi\)-separable group. We consider the Isaacs \(B_\pi\)-characters. We show that if \(N\) is a normal subgroup of \(G\), then \(B_\pi(G/N) = \text{Irr}(G/N) \cap B_\pi(G)\).

1. Introduction

All groups in this paper are finite. Throughout this paper \(\pi\) will be a set of primes and \(G\) will be a \(\pi\)-separable group. In [3], Isaacs defined the subset \(B_\pi(G)\) of \(\text{Irr}(G)\). In our paper [6] about a variation on Landau’s theorem, we needed a basic fact about the characters in \(B_\pi(G)\) and quotients of \(G\) that seems not to have been proved anywhere. Since proving this fact in [6] would have been a distraction to the main point of that paper, we decided to establish the proof of this fact separately.

Many of the basic ideas about the set \(B_\pi(G)\) were proved in [3], and in fact, all of the facts we need about \(B_\pi(G)\) can be found in [3]. The papers [4] and [5] both give very good expository accounts about \(B_\pi(G)\) characters. Much of this paper is to provide the terminology, concepts, and definitions needed to explicitly define the set \(B_\pi(G)\).

Suppose \(G\) is a group and \(N\) is a normal subgroup of \(G\). There is a bijection between the sets \(\{\chi \in \text{Irr}(G) \mid N \leq \ker(\chi)\}\) and \(\text{Irr}(G/N)\) (see Lemma 2.22 of [2]). If \(\chi \in \text{Irr}(G)\) and \(N \leq \ker(\chi)\), then we write \(\hat{\chi}\) to denote the corresponding character in \(\text{Irr}(G/N)\). With this in mind, we set the following notation: \(\hat{B}_\pi(G) = \{\hat{\chi} \mid \chi \in B_\pi(G), N \leq \ker(\chi)\}\). Notice that if we identify \(\text{Irr}(G/N)\) and \(B_\pi(G/N)\) with the appropriate subsets of \(\text{Irr}(G)\), then \(\hat{B}_\pi(G) = \text{Irr}(G/N) \cap B_\pi(G)\).

The goal of this paper is to prove the following:


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Theorem 1.1. Suppose \( \pi \) is a set of primes and \( G \) is a \( \pi \)-separable group. If \( N \) is a normal subgroup of \( G \), then \( B_\pi(G/N) = \hat{B}_\pi(G) \).

2. Results

The first idea that we need for defining the set \( B_\pi(G) \) are \( \pi \)-special characters. Following Gajendragadkar in [1], we say that a character \( \chi \in \text{Irr}(G) \) where \( G \) is \( \pi \)-special if \( \chi(1) \) is a \( \pi \)-number and for every subnormal subgroup \( S \) of \( G \), the irreducible constituents of \( \chi_S \) have determinantal order that is a \( \pi \)-number.

In Proposition 7.1 of [1], Gajendragadkar proved that if \( \alpha, \beta \in \text{Irr}(G) \) are characters so that \( \alpha \) is \( \pi \)-special and \( \beta \) is \( \pi' \)-special, then \( \alpha \beta \) is irreducible, and this factorization is unique. i.e., if \( \alpha \beta = \alpha' \beta' \) where \( \alpha' \) is \( \pi \)-special and \( \beta' \) is \( \pi' \)-special, then \( \alpha = \alpha' \) and \( \beta = \beta' \).

Using [3], we say that \( \chi \in \text{Irr}(G) \) is \( \pi \)-factored if there exists a \( \pi \)-special character \( \alpha \) and \( \pi' \)-special character \( \beta \) so that \( \chi = \alpha \beta \). We take time out of our definitions to prove the following lemma regarding the kernels of \( \pi \)-factored characters which is key to our argument.

Lemma 2.1. Suppose \( \pi \) is a set of primes and \( G \) is a \( \pi \)-separable group. If \( \chi \in \text{Irr}(G) \) satisfies \( \chi = \alpha \beta \) where \( \alpha \) is \( \pi \)-special and \( \beta \) is \( \pi' \)-special, then \( \ker(\chi) = \ker(\alpha) \cap \ker(\beta) \).

Proof. It is obvious that \( \ker(\alpha) \cap \ker(\beta) \leq \ker(\chi) \leq K \). We need to show that \( K \leq \ker(\alpha) \cap \ker(\beta) \). We first claim that \( K \leq Z(\alpha) \cap Z(\beta) \). Suppose \( g \in K \). Then \( \alpha(g)\beta(g) = \chi(g) = \chi(1) = \alpha(1)\beta(1) \). Hence, \( \alpha(1)\beta(1) = |\alpha(g)\beta(g)| = |\alpha(g)||\beta(g)| \). By Lemma 2.15(c) of [2], we know that \( |\alpha(g)| \leq \alpha(1) \) and \( |\beta(g)| \leq \beta(1) \). The previous equality implies that these inequalities must be equalities, so \( g \in Z(\alpha) \) and \( g \in Z(\beta) \). This proves the claim.

By Lemma 2.27 (c) of [2], we see that \( \alpha_K = \alpha(1)\mu \) and \( \beta_K = \beta(1)\nu \) for linear characters \( \mu \) and \( \nu \) in \( \text{Irr}(K) \). Because \( \alpha \) is \( \pi \)-special, \( \mu \) must have \( \pi \)-order and because \( \beta \) is \( \pi' \)-special, \( \nu \) must have \( \pi' \)-order. For \( g \in K \), this implies that \( \mu(g) \) is a \( \pi \) root of unity and \( \nu(g) \) is a \( \pi' \) root of unity. We have \( \alpha(1)\beta(1) = \chi(1) = \chi(g) = \alpha(g)\beta(g) = \alpha(1)\mu(1)\beta(1)\nu(g) \). This implies that \( \nu(g)\mu(g) = 1 \). The only way that the product of a \( \pi \) root of unity and a \( \pi' \) root can equal 1 is if they are both 1. i.e., we must have \( \mu(g) = \nu(g) = 1 \). This implies that \( \alpha(1) = \alpha(g) \) and \( \beta(1) = \beta(g) \). Therefore, \( g \in \ker(\alpha) \cap \ker(\beta) \) as desired.

Returning to our definitions, we fix the character \( \chi \in \text{Irr}(G) \). We say that \( (S, \sigma) \) is a subnormal pair for \( \chi \) if \( S \) is a subnormal subgroup of \( G \) and \( \sigma \) is an irreducible constituent of \( \chi_S \). In addition, we say that \( (S, \sigma) \) is \( \pi \)-factored if \( \sigma \) is \( \pi \)-factored. We can define a partial ordering on the subnormal pairs for \( \chi \) by \( (S, \sigma) \leq (T, \tau) \) if \( S \leq T \) and \( \sigma \) is a constituent of \( \tau_S \).

Notice that \( (1, 1_1) \) is a \( \pi \)-factored subnormal pair for \( \chi \), so there exists a maximal \( \pi \)-factored subnormal pair for \( \chi \) with respect to the partial ordering. It is shown in Theorem 3.2 of [3] that the set of maximal \( \pi \)-factored subnormal pairs for \( \chi \) are conjugate in \( G \). Let \( (S, \sigma) \) be a maximal \( \pi \)-factored subnormal pair for \( \chi \), and let \( T \) be the stabilizer of \( (S, \sigma) \) in \( G \). It is shown in Theorem 4.4 of [3] that there is a unique character \( \tau \in \text{Irr}(T | \sigma) \) so that \( \tau_G^G = \chi \).
We can now define the $\pi$-nucleus for $\chi$. If $\chi$ is $\pi$-factored, then $(G, \chi)$ is the nucleus for $\chi$. If $\chi$ is not $\pi$-factored, then let $(S, \sigma)$ be a maximal $\pi$-factored subnormal pair for $\chi$. Let $T$ be the stabilizer of $(S, \sigma)$ in $G$, and let $\tau \in \text{Irr}(T \mid \sigma)$ so that $\tau^G = \chi$. By Lemma 4.5 of [3], we know that $T < G$, so we can inductively define the $\pi$-nucleus of $\chi$ to be the $\pi$-nucleus of $\tau$. Because the maximal $\pi$-factored subnormal pairs are all conjugate, it follows that the $\pi$-nucleus for $\chi$ is well-defined up to conjugacy. (See the argument on page 108 of [3].)

We are now ready to state the definition of $B_\pi(G)$. We still have the character $\chi$. We take $(X, \eta)$ to be a $\pi$-nucleus for $\chi$, and by definition $\eta$ must be $\pi$-factored. The set $B_\pi(G)$ is defined to be those characters $\chi \in \text{Irr}(G)$ where $(X, \eta)$ is a $\pi$-nucleus for $\chi$ and $\eta$ is $\pi$-special. This is the statement of Definition 5.1 of [3].

We now give a lemma that connects a $\pi$-nucleus with the quotient.

**Lemma 2.2.** Let $\pi$ be a set of primes and let $G$ be a $\pi$-separable group. Suppose that $N$ is a normal subgroup of $G$. If $\chi \in \text{Irr}(G)$ with $N \leq \ker(\chi)$ has $\pi$-nucleus $(X, \eta)$, then $(X/N, \hat{\eta})$ is a $\pi$-nucleus for $\hat{\chi} \in \text{Irr}(G/N)$.

**Proof.** If $(X, \eta) = (G, \chi)$, then this is obvious. Thus, we may assume that $X < G$. Let $(S, \sigma)$ be a maximal $\pi$-factored subnormal pair for $\chi$ with stabilizer $T$ and character $\tau \in \text{Irr}(T \mid \sigma)$ so that $\tau^G = \chi$ and $(X, \eta)$ is a $\pi$-nucleus for $\tau$. Notice that $(N, 1_N)$ is a $\pi$-factored subnormal pair for $\chi$, so it is contained in a maximal such pair. As we mentioned above, the maximal $\pi$-factored subnormal pairs for $\chi$ are all conjugate. Since $N$ is normal, this implies that $N \leq S$, and thus, $(N, 1_N) \leq (S, \sigma)$. Because $\sigma$ is a constituent of $\chi_S$, we see that $N \leq \ker(\sigma)$. By Lemma 2.1, we see that $\hat{\sigma}$ is $\pi$-factored as a character in $\text{Irr}(S/N)$. Notice that $(S/N, \hat{\sigma}) \leq (S^\ast/N, \hat{\sigma}^\ast)$ if and only if $(S, \sigma) \leq (S^\ast, \sigma^\ast)$, and by Lemma 2.1, $\hat{\sigma}^\ast$ is $\pi$-factored in $\text{Irr}(S^\ast/N)$ if and only if $\sigma$ is $\pi$-factored in $\text{Irr}(S^\ast)$. Therefore, $(S/N, \hat{\sigma})$ must be a maximal $\pi$-factored subnormal pair for $\hat{\chi}$. It is immediate that $T/N$ will be the stabilizer for $(S/N, \hat{\sigma})$ in $G/N$ and that $\hat{\tau}$ is the unique character in $\text{Irr}(T/N \mid \hat{\sigma})$ that induces $\hat{\chi}$. By induction, $(X/N, \hat{\eta})$ will the $\pi$-nucleus for $\hat{\tau}$, and thus, $(X/N, \hat{\eta})$ will be a $\pi$-nucleus for $\hat{\chi}$. □

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Note that $B_\pi(G/N) \subseteq \text{Irr}(G/N)$. Hence, it suffices to show for $\chi \in \text{Irr}(G)$ with $N \leq \ker(\chi)$ that $\chi \in B_\pi(G)$ if and only if $\hat{\chi} \in B_\pi(G/N)$. Suppose $\chi \in \text{Irr}(G/N)$. Let $(X, \eta)$ be a $\pi$-nucleus for $\chi$. By Lemma 2.2, $(X/N, \hat{\eta})$ is a nucleus for $\hat{\chi}$. We know that $\chi \in B_\pi(G)$ if and only if $\eta$ is $\pi$-special and $\hat{\chi} \in B_\pi(G/N)$ if and only if $\hat{\eta}$ is $\pi$-special. Using the definition of $\pi$-special, we see that $\eta$ is $\pi$-special if and only if $\hat{\eta}$ is $\pi$-special, and this proves the theorem. □

**References**


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