www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. x No. x (201x), pp. xx-xx.

www.ui.ac.ir

ON ALGEBRAIC GEOMETRY OVER COMPLETELY SIMPLE SEMIGROUPS

ARTEM N. SHEVLYAKOV

Communicated by Evgeny Vdovin

Abstract. We study equations over completely simple semigroups and describe the coordinate semigroups of irreducible algebraic sets for such semigroups.

1. Introduction

Many problems of semigroup theory are reduced to the similar problems of group theory. This approach is often used for completely simple (c.s.) semigroups, since any c.s. semigroup G is isomorphic to a disjoint union of isomorphic copies of a group G (G is called the structural group of S). Thus, S actually inherits a big part its properties from G, and one can formulate the general principle:

$$S \text{ has a property } P \Leftrightarrow G \text{ has a property } P. \tag{1.1}$$

In this paper we study equations over c.s. semigroups. Remark that the equations over c.s. semigroups have more complicated view than group equations. It turns out that the principle (1.1) also holds for equations over c.s. semigroups. For example, it was proved in [6] that the property $P=\{$to be an equationally Noetherian semigroup$\}$ satisfies (1.1).

Below we describe the coordinate semigroups of irreducible algebraic sets over a c.s. semigroup S. We show that this problem is reduced to the same problem for the structural group G of S (see Theorems 3.1, 5.1). In Theorem 3.1 we study equations with no constants, whereas in Theorem 5.1 we consider equations with constants.

MSC(2010): Primary: 20M17; Secondary: 20M99.

Keywords: system of equations, coordinate semigroups, universal algebraic geometry.

Received: 16 April 2016, Accepted: 01 September 2017.

Moreover, in the paper we deal with free semigroups of the variety of c.s. semigroups with abelian structural groups and prove that such semigroups are geometrically equivalent to each other (Theorem 4.2).

2. Basic Notions

The class of completely simple (c.s.) semigroups is one of the important semigroup classes. The c.s. semigroups are described by the following famous theorem.

Theorem 2.1. *For any c.s. semigroup S there exists a group G and sets I,Λ such that S is isomorphic to the set of triples (λ,g,i), $g\in G$, $\lambda\in\Lambda$, $i\in I$. The multiplication over the triples (λ,g,i) is defined as follows*

$$(\lambda,g,i)(\mu,h,j)=(\lambda,gp_{i\mu}h,j),$$

where $p_{i\mu}\in G$ is an element of a matrix $\mathbf{P}$ such that

1. *$\mathbf{P}$ consists of $|I|$ rows and $|\Lambda|$ columns;*
2. *the matrix $\mathbf{P}$ is (λ,i)-normalised for some $\lambda\in\Lambda$, $i\in I$, i.e. all elements of the λ-th row and i-th column equal $1\in G$.*

Following Theorem 2.1, we denote any c.s. semigroup S by $S=(G,\mathbf{P},\Lambda,I)$. The group G and the matrix $\mathbf{P}$ are called the *structural group* and the *sandwich-matrix* respectively. The elements λ,i occurring in a triple $(\lambda,g,i)\in S$ are *the first and the second indexes* respectively. We will call the set $R_\lambda=\{(\lambda,g,i)\mid i\in I\}\subseteq S$ ($C_i=\{(\lambda,g,i)\mid\lambda\in\Lambda\}\subseteq S$) the *$\lambda$-th row* (respectively, *i-th column*) of S. Formally, each row (column) of a c.s. semigroup S is a minimal right (respectively, left) ideal of S.

Remark 2.2. *We implicitly use below that the following transformations of a c.s. semigroup S with a presentation $(G,\mathbf{P},\Lambda,I)$ do not change the semigroup structure of S:*

1. *any substitution $G\mapsto G'$, where a group G' is isomorphic to G;*
2. *a multiplication of any row (column) of $\mathbf{P}$ by an element $g\in G$ (i.e. one can normalize $\mathbf{P}$ for any row and column).*
3. *any swap of rows (columns) in the sandwich-matrix $\mathbf{P}$.*

Let $\langle\mathbf{P}\rangle\subseteq G$ denote the group generated by all entries of the sandwich-matrix $\mathbf{P}$.

By Theorem 2.1, any c.s. semigroup S is the disjoint union of copies of the structural group G, i.e. all maximal subgroups of S are isomorphic to G. Clearly, the identity elements of the maximal subgroups are $(\lambda,{p_{i\lambda}}^{-1},i)$, $\lambda\in\Lambda$, $i\in I$. The inversion $^{-1}$ in a subgroup defined by the indexes λ,i is

$$(\lambda,g,i)^{-1}=(\lambda,{p_{i\lambda}}^{-1}g^{-1}{p_{i\lambda}}^{-1},i).$$

The class of c.s. semigroups is a variety in the language $\{\cdot,{}^{-1}\}$, since it is defined by the identities:

$$xx^{-1}x=x,\ xx^{-1}=x^{-1}x,\ (x^{-1})^{-1}=x,\ (xyx)^{-1}(xyx)=x^{-1}x.$$

The structure of the free c.s. semigroups is described by the following theorem.

Theorem 2.3. [2, 9] *Let* $X_n = \{x_1, x_2, \dots, x_n\}$, $Y_n = \{y_{i\lambda} | 2 \leq i \leq n, 2 \leq \lambda \leq n\}$ *be finite sets of letters, and* $I_n = \Lambda_n = \{1, 2, \dots, n\}$. *Let* $F(X_n \cup Y_n)$ *denote the free group generated by the set* $X_n \cup Y_n$. *Thus, the free c.s. semigroup* $\mathcal{F}_n$ *of rank* n *is defined by* $\mathcal{F}_n = (F(X_n \cup Y_n), \mathbf{P}_n, I_n, \Lambda_n)$, *where* $\mathbf{P}_n = (p_{i\lambda} | i \in I_n, \lambda \in \Lambda_n)$,

$$p_{i\lambda} = \begin{cases} 1, & \text{if } i = 1 \text{ or } \lambda = 1, \\ y_{i\lambda}, & \text{otherwise} \end{cases}$$

and the free generators x_i *correspond to the triples* $(i, x_i, i) \in \mathcal{F}_n$.

Let $\mathbf{CS}_{ab}$ be the class of c.s. semigroups with abelian structural groups. Following [10], $\mathbf{CS}_{ab}$ is a variety, and Theorem 2.3 gives the structure of the free semigroup $\mathcal{F}_n^{ab} \in \mathbf{CS}_{ab}$ of rank n. Indeed,

$$\mathcal{F}_n^{ab} = (F_N, \mathbf{P}_n, \Lambda_n, I_n), \tag{2.1}$$

where F_N is isomorphic to the free abelian group generated by $X_n \cup Y_n$ (the sets X_n, Y_n, Λ_n, I_n and the sandwich-matrix $\mathbf{P}_n$ were defined in Theorem 2.3). Thus, the structural group of $\mathcal{F}_n^{ab}$ is isomorphic to the direct sum

$$\bigoplus_{1 < \lambda, i \leq n} \mathbb{Z}_{\lambda i} \oplus \mathbb{Z}^n = \langle \mathbf{P} \rangle \oplus \mathbb{Z}^n, \tag{2.2}$$

where $\mathbb{Z}_{\lambda i}$ are isomorphic copies of the group $\mathbb{Z}$.

The following result describes the set of homomorphisms of c.s. semigroups.

Theorem 2.4. [3] *Let* $S = (G, \mathbf{P}, \Lambda, I)$, $S' = (G', \mathbf{P}', \Lambda', I')$ *be a c.s. semigroups and* $\varphi \in \mathrm{Hom}(S, S')$. *Then, there exist mappings*

(1) $\bar{h} \colon I \to G'$, $\bar{\chi} \colon \Lambda \to G'$,
(2) $h \colon I \to I'$, $\chi \colon \Lambda \to \Lambda'$,
(3) *a group homomorphism* $\omega \colon G \to G'$

such that

$$\omega(p_{i\lambda}) = \bar{\chi}(\lambda) p_{h(i)\chi(\lambda)} \bar{h}(i) \tag{2.3}$$

and

$$\varphi((\lambda, g, i)) = (\chi(\lambda), \bar{h}(i)\omega(g)\bar{\chi}(\lambda), h(i)). \tag{2.4}$$

Lemma 2.5. *Let* $S = (G, \mathbf{P}, \Lambda, I)$, $S' = (G', \mathbf{P}', \Lambda', I')$ *be c.s. semigroups with* $(1,1)$*-normalized* $\mathbf{P}, \mathbf{P}'$. *Suppose a homomorphism* $\varphi \colon S \to S'$ (2.4) *maps the first column of* S *into the first column in* S', *and the first row of* S *is mapped into the row column in* S'. *Then there exists an element* $a \in G'$ *such that* $\bar{h}(i) = a$, $\bar{\chi}(\lambda) = a^{-1}$ *for all* $i \in I$, $\lambda \in \Lambda$, *and*

$$\omega(p_{i\lambda}) = a^{-1} p_{h(i)\chi(\lambda)} a. \tag{2.5}$$

Proof. By (2.3), for any $i \in I$, $\lambda \in \Lambda$ we have

$$1 = \omega(1) = \omega(p_{1\lambda}) = \bar{\chi}(\lambda)p_{h(1)\chi(\lambda)}\bar{h}(1) = \bar{\chi}(\lambda)p_{1\chi(\lambda)}\bar{h}(1) = \bar{\chi}(\lambda)\bar{h}(1),$$

$$1 = \omega(1) = \omega(p_{i1}) = \bar{\chi}(1)p_{h(i)\chi(1)}\bar{h}(i) = \bar{\chi}(1)p_{h(i)1}\bar{h}(i) = \bar{\chi}(1)\bar{h}(i).$$

In particular, $\bar{\chi}(1)\bar{h}(1) = 1$. Let $a = h(1)$, then $a^{-1} = \bar{\chi}(\lambda)$, $a = \bar{h}(i)$, and we immediately obtain (2.5). □

Let us give the basic notions of universal algebraic geometry. All definitions below are derived from the general notions of [4], where such definitions were formulated for an arbitrary algebraic structure in the language with no predicates.

We consider c.s. semigroups (as algebraic structures) in the language $\mathcal{L} = \{\cdot, ^{-1}\}$ with natural interpretation of the functional symbols $\cdot, ^{-1}$. Let X be a finite set of variables $x_1, x_2, \ldots, x_n$. *A term* of a language $\mathcal{L}$ ($\mathcal{L}$-term) in variables X is one of the following expressions:

(1) a variable x_i;
(2) a product $t(X)s(X)$ of two $\mathcal{L}$-terms $t(X), s(X)$;
(3) $(t(X))^{-1}$, where $t(X)$ is a $\mathcal{L}$-term.

For example, the expressions $((x^{-1})^{-1})^{-1}$ $(xy)^{-1}$, $x(yz^{-1})^{-1}$ are $\mathcal{L}$-terms.

An equation over $\mathcal{L}$ (*$\mathcal{L}$-equation*) is an equality of two $\mathcal{L}$-terms $t(X) = s(X)$. *A system of equations* over $\mathcal{L}$ (*$\mathcal{L}$-system* or *system* for shortness) is an arbitrary set of $\mathcal{L}$-equations.

A point $P = (p_1, p_2, \ldots, p_n) \in S^n$ is a *solution* of a system $\mathbf{S}$ in variables $x_1, x_2, \ldots, x_n$ if the substitution $x_i = p_i$ reduces any equation of $\mathbf{S}$ to a true equality in a c.s. semigroup S. The set of all solutions of a system $\mathbf{S}$ in a c.s. semigroup S is denoted by $\mathrm{V}_S(\mathbf{S})$. A set $Y \subseteq S^n$ is called *algebraic* over a c.s. semigroup S if there exists a $\mathcal{L}$-system in variables $x_1, x_2, \ldots, x_n$ with the solution set Y. A nonempty algebraic set Y is *irreducible* if it is not a proper finite union of other algebraic sets.

Let us give examples of algebraic sets over a c.s. semigroup S.

(1) $Y = \{(x, y) \mid x, y \text{ belong to the same maximal subgroup}\} = \mathrm{V}_S(xx^{-1} = yy^{-1})$.
(2) $Y = \{(x, y) \mid x, y \text{ belong to the same row}\} = \mathrm{V}_S((xy)(xy)^{-1} = yy^{-1})$.

Let Y be a non-empty algebraic set defined by a system of $\mathcal{L}$-equations. Let us define the equivalence relation over the set of $\mathcal{L}$-terms $\mathcal{T}_S$ by

$$t(X) \sim_Y s(X) \Longleftrightarrow t(P) = s(P) \text{ for any } P \in Y.$$

Denote the factor-semigroup $\mathcal{T}_S/\sim_Y$ by $\Gamma_S(Y)$. One can prove that $\Gamma_S(Y)$ is a c.s. semigroup. Following [4], $\Gamma_S(Y)$ is called *the coordinate semigroup of an algebraic set Y*, and the *main aim of universal algebraic geometry is the description of the coordinate semigroups* of algebraic sets over a given semigroup S.

Theorem 2.6. [5] *Let S, S' be a completely simple semigroups and S is finitely generated. Then the following conditions are equivalent:*

(1) *S is the coordinate semigroup of an irreducible algebraic set over S';*

(2) *S is discriminated by S' (i.e. for any finite set $\{s_1, s_2, \dots, s_m\} \subseteq S$ there exist a homomorphism $\varphi\colon S \to S'$ with $\varphi(s_i) \neq \varphi(s_j)$ for all $i \neq j$).*

Suppose c.s. semigroups S, S' contain a subsemigroup T. We say that *S is T-discriminated by S'* if for any finite set $\{s_1, s_2, \dots, s_m\} \subseteq S$ there exist a homomorphism $\varphi\colon S \to S'$ with $\varphi(s_i) \neq \varphi(s_j)$ for all $i \neq j$ and $\varphi(t) = t$ for each $t \in T$.

3. Coordinate Semigroups

Theorem 3.1. *Let $S' = (G', \mathbf{P}', \Lambda', I')$. A finitely generated semigroup $S = (G, \mathbf{P}, \Lambda, I)$ is the coordinate semigroup of an irreducible algebraic set over S' iff there exists a subsemigroup $T' \subseteq S'$ isomorphic to $(G', \mathbf{P}, \Lambda, I)$, and G' $\langle\mathbf{P}\rangle$-discriminates G.*

Proof. Let us prove the "only if" part of the theorem. Without loss of generality one can assume that the sandwich matrix $\mathbf{P}$ of S is $(1,1)$-normalized. Let E be the set of all idempotents of S (since S is finitely generated, E is finite), and G_0 an arbitrary finite subset in G

$$P = \{(1, p, 1) \mid p \text{ occurs in the sandwich-matrix } \mathbf{P}\},$$

$$\Gamma = \{(1, g, 1) \mid g \in G_0\}.$$

By Theorem 2.6, S is discriminated by S', so there exists a homomorphism $\varphi\colon S \to S'$ injective on $E \cup P \cup \Gamma$. Let us denote the image of φ by $\bar{T}$. Following Remark 2.2, one can assume that $\bar{T}$ is isomorphic to the semigroup $(\bar{G}, \bar{\mathbf{P}}, \bar{\Lambda}, \bar{I})$, where

(1) the sandwich-matrix $\bar{\mathbf{P}}$ is $(1,1)$-normalized;
(2) φ maps the first column of S into the first column of $\bar{T}$ and the first row of S is mapped into the first row of $\bar{T}$.

By Lemma 2.5, it holds (2.5) for some $a \in \bar{G}$.

Since φ is injective on the set of all idempotents $E \subseteq S$, we have $\bar{\Lambda} = \Lambda$, $\bar{I} = I$. One can renumber the rows and columns of $\bar{T}$ and the mappings h, χ of the homomorphism φ will satisfy $h(i) = i$, $\chi(\lambda) = \lambda$.

Clearly, the element a^{-1} defines the inner automorphism $g^{a^{-1}} = aga^{-1}$ of the group $\bar{G}$. Let $\bar{G}^a$ denote the image of $\bar{G}$ via this inner automorphism. Applying the inner automorphism to the elements of sandwich-matrix $\bar{\mathbf{P}}$, we obtain a new matrix $\bar{\mathbf{P}}^a$.

Let T^a denote the c.s. semigroup $(\bar{G}^a, \bar{\mathbf{P}}^a, \Lambda, I)$, and $\psi\colon \bar{T} \to T^a$ is an isomorphism between $\bar{T}, T^a$. Then for the homomorphism $\phi = \psi \circ \varphi$ the equality (2.5) becomes

$$\psi(\omega(p_{i\lambda})) = p_{h(i)\chi(\lambda)} = p_{i\lambda}. \tag{3.1}$$

Therefore, $\bar{\mathbf{P}}^a = \mathbf{P}$, and we have $T^a = (\bar{G}^a, \mathbf{P}, \Lambda, I)$. Since we arbitrarily chose the set Γ, the group $\bar{G}^a$ discriminates G and by (3.1) ϕ does not change elements of $\mathbf{P}$. Thus, the group $\bar{G}^a$ $\langle\mathbf{P}\rangle$-discriminates G. Since $T^a \subseteq T' = (G', \mathbf{P}, \Lambda, I) \subseteq S'$ and $\bar{G}^a$ is embedded into G', G' $\langle\mathbf{P}\rangle$-discriminates G.

Now we prove the "if" part of the theorem. Let $a_1, a_1, \dots, a_n$ be distinct elements of S with $a_i = (\lambda_i, g_i, k_i)$.

By condition, there exists a group $\langle\mathbf{P}\rangle$-homomorphism $\phi\colon G \to G'$ with $\phi(g_i) \neq \phi(g_j)$ for any $g_i \neq g_j$.

Define a map $\psi\colon S \to T$ by $\psi((\lambda, g, k)) = (\lambda, \phi(g), k)$.

The map ψ is a homomorphism, since

$$\psi((\lambda, g, k)(\mu, h, l)) = \psi((\lambda, gp_{k\mu}h, l)) = (\lambda, \phi(gp_{k\mu}h), l) = (\lambda, \phi(g)\phi(p_{k\mu})\phi(h), l) = (\lambda, \phi(g)p_{k\mu}\phi(h), l),$$

and

$$\psi((\lambda, g, k))\psi((\mu, h, l)) = (\lambda, \phi(g), k)(\mu, \phi(h), l) = (\lambda, \phi(g)p_{k\mu}\phi(h), l).$$

For the elements a_i, a_j $(i \neq j)$ we have

$$\psi(a_i) = (\lambda_i, \phi(g_i), k_i),\ \psi(a_i) = (\lambda_j, \phi(g_j), k_j).$$

If $\lambda_i \neq \lambda_j$ or $k_i \neq k_j$ it follows $\psi(a_i) \neq \psi_i(a_j)$. Otherwise $(g_i \neq g_j)$, by the choice of ϕ we obtain $\phi(g_i) \neq \phi(g_j)$, hence $\psi(a_i) \neq \psi_i(a_j)$. □

Using Theorem 3.1, one can obtain the description of the coordinate semigroups of irreducible algebraic sets over $\mathcal{F}_n^{ab}$ for $n \geq 2$.

Theorem 3.2. *A finitely generated c.s. semigroup S is the coordinate semigroup of an irreducible algebraic set over $\mathcal{F}_n^{ab}$ $(n \geq 2)$ iff it is isomorphic to a c.s. $(G, \mathbf{P}, \Lambda, I)$, where*

(1)
$$\Lambda = \{1, 2, \dots, l\},\ I = \{1, 2, \dots, m\},\ l \leq n,\ m \leq n$$

(2)
$$G = \bigoplus_{i \in I, \lambda \in \Lambda} \mathbb{Z}_{i\lambda} \oplus \mathbb{Z}^k \tag{3.2}$$

for some $k \geq 0$ ($\mathbb{Z}_{i\lambda}$ is the isomorphic copy of the infinite cyclic group $\mathbb{Z}$).

(3) $\mathbf{P} = (p_{i\lambda}) = (e_{i\lambda})$, *where $e_{i\lambda}$ is the generating element of the maximal subgroup $\mathbb{Z}_{i\lambda} \subseteq S$. In other words, $G = \langle\mathbf{P}\rangle \oplus \mathbb{Z}^k$.*

Proof. Suppose $\mathcal{F}_n^{ab}$ has a presentation (2.1), and S is the coordinate semigroup of an irreducible set over $\mathcal{F}_n^{ab}$. By Theorem 3.1, $\mathbf{P}$ is embedded into $\mathbf{P}_n$, $\Lambda \subseteq \Lambda_n$, $I \subseteq I_n$ and $\mathbf{P} = (p_{i\lambda}) = (e_{i\lambda})$ $(i \in I, \lambda \in \Lambda)$.

Any group $\langle\mathbf{P}\rangle$-discriminated by F_N is of the form $\langle\mathbf{P}\rangle \oplus \mathbb{Z}^k$ (see [8]), and we obtain that S has the desired presentation $(G, \mathbf{P}, \Lambda, I)$.

Let us prove the "if" statement of the theorem. Let $S = (G, \mathbf{P}, \Lambda, I)$ with $G, \mathbf{P}, \Lambda, I$ defined above. Obviously, $\mathbf{P}$ is embedded into $\mathbf{P}_n$. By Theorem 3.1, it is sufficient to prove G is $\langle\mathbf{P}\rangle$-discriminated by F_N. By condition, $G = \langle\mathbf{P}\rangle \oplus \mathbb{Z}^k$, $F_N = \langle\mathbf{P}\rangle \oplus \mathbb{Z}^n$. The group $\mathbb{Z}^k$ is discriminated by $\mathbb{Z}^n$ (see [8]), and any homomorphism $\psi \in \mathrm{Hom}(\mathbb{Z}^k, \mathbb{Z}^n)$ can be extended to a homomorphism $\psi' \in \mathrm{Hom}(G, F_N)$ which acts identically on $\langle\mathbf{P}\rangle$. Thus, G is $\langle\mathbf{P}\rangle$-discriminated by F_N. □

4. Algebraic Geometry over the Free c.s. Semigroups

In this section we show that all free semigroups $\{\mathcal{F}_n^{ab} \mid n \geq 2\}$ with abelian structural groups generate the same algebraic geometry. Let us formalize this assertion as follows. Two c.s. semigroups S, S' are *geometrically equivalent* if for any $\mathcal{L}$-system $\mathbf{S}$ the coordinate semigroups $\Gamma_S(\mathrm{V}_S(\mathbf{S}))$, $\Gamma_{S'}(\mathrm{V}_{S'}(\mathbf{S}))$ are isomorphic. It follows that the geometrically equivalent semigroup S, S' have the same set of the coordinate algebras of algebraic sets.

We shall use the following test of geometrical equivalence (this result directly follows from Unifying theorems of [4]).

Theorem 4.1. *Finitely generated c.s. semigroups S, S' are geometrically equivalent iff S is separated by S' (i.e. for any distinct elements $s_1, s_2 \in S$ there exists a homomorphism $\varphi\colon S \to S'$ with $\varphi(s_1) \neq \varphi(s_2)$) and S' is separated by S.*

Remark that the semigroups $\mathcal{F}_n^{ab}$ satisfy the condition of Theorem 4.1, since they are finitely generated.

Let us define a completely simple semigroup $\mathcal{F}_0$ with a structural group $\mathbb{Z} = \langle 1 \rangle$ and a sandwich-matrix

$$\mathbf{P}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem 4.2. *The semigroups $\mathcal{F}_n^{ab}, \mathcal{F}_m^{ab}$ are geometrically equivalent for any $m, n \geq 2$.*

Proof. Obviously, it is sufficient to prove that $\mathcal{F}_n^{ab}$ is geometrically equivalent to $\mathcal{F}_0$. According to Theorem 4.1, we prove that $\mathcal{F}_n^{ab}, \mathcal{F}_0$ separate each other.

Obviously $\mathcal{F}_0$ is separated by $\mathcal{F}_n^{ab}$, since it is embedded into $\mathcal{F}_n^{ab}$ for each $n \geq 2$. Let us prove the converse.

According to (2.2), the structural group and the sandwich-matrix of $\mathcal{F}_n^{ab}$ are

$$F_N = \langle \mathbf{P} \rangle \oplus \mathbb{Z}^n,$$

and $\mathbf{P}_n = (p_{i\lambda}) = (e_{i\lambda})$, where $e_{i\lambda}$ is the generator of the subgroup $\mathbb{Z}_{i\lambda} \cong \mathbb{Z}$. Obviously,

$$\langle \mathbf{P}_n \rangle = \bigoplus_{1 \leq \lambda, i \leq n} \mathbb{Z}_{i\lambda}.$$

Let $s_1 = (\lambda, g, i)$, $s_2 = (\mu, h, i)$ be two distinct elements of $\mathcal{F}_n^{ab}$. Let us define a homomorphism $\psi\colon \mathcal{F}_n^{ab} \to \mathcal{F}_0$ with $\psi(s_1) \neq \psi(s_2)$.

Consider the next three cases.

(1) Let $\lambda \neq \mu$ and define ψ as follows

$$\psi((\nu, f, k)) = \begin{cases} (1, 0, 1) \text{ if } \nu \neq \mu, \\ (2, 0, 1) \text{ if } \nu = \mu \end{cases}$$

Roughly speaking, ψ maps all columns of the Rees matrix semigroup $\mathcal{F}_n^{ab}$ into the first column of $\mathcal{F}_0$ and all rows with indexes not equal to μ are mapped into the first row of $\mathcal{F}_0$.

It is easy to check that $\psi \in \mathrm{Hom}(\mathcal{F}_n^{ab}, \mathcal{F}_0)$ and $\psi(s_1) \neq \psi(s_2)$, since s_1, s_2 are mapped into the first and second rows respectively.

(2) The case $i \neq j$ is similar to the previous one.

(3) Suppose $\lambda = \mu$ and $i = j$. The elements g, h are uniquely represented as

$$g = p_1 + g',\ h = p_2 + h',$$

where $g', h' \in \mathbb{Z}^n$, $p_1, p_2 \in \langle \mathbf{P}_n \rangle$.

We have exactly two cases:

(a) Suppose $g' \neq h'$. Since $\mathbb{Z}^n$ is obviously separated by $\mathbb{Z}$, there exists a homomorphism $\varphi \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ with $\varphi(g') \neq \varphi(h')$. Let φ' be the zero-extension of φ to F_N.
Thus, $\varphi'(g) \neq \varphi'(h)$. Define a map ψ by

$$\psi((\nu, f, k)) = (1, \varphi'(f), 1).$$

It is directly checked that $\psi \in \mathrm{Hom}(\mathcal{F}_n^{ab}, \mathcal{F}_0)$ and $\psi(s_1) \neq \psi(s_2)$.

(b) Assume now that $g' = h'$ and $p_1 \neq p_2$. The elements p_1, p_2 equal to the sums

$$p_j = \sum_{i,\lambda} p_{ji\lambda},\ (1 \leq j \leq 2),$$

where $p_{ji\lambda} \in \mathbb{Z}_{i\lambda}$.
By the condition, there exists a pair of indexes $i' \in I_n$, $\lambda' \in \Lambda_n$ such that $p_{1i'\lambda'} \neq p_{2i'\lambda'}$.
The homomorphism $\varphi \colon \langle \mathbf{P} \rangle \to \mathbb{Z}$ by

$$\varphi(x) = \begin{cases} x, \text{ if } x \in \mathbb{Z}_{i'\lambda'} \\ 0, \text{ otherwise} \end{cases}$$

has the property $\varphi(g) \neq \varphi(h)$.
Let φ' be the zero-extension of φ to F_N. Define a map $\psi \colon \mathcal{F}_n^{ab} \to \mathcal{F}_0$ by

$$\psi((\nu, f, k)) = \begin{cases} (2, \varphi'(f), 2) \text{ if } \nu = \lambda' \text{ and } k = i', \\ (2, 0, 1) \text{ if } k = i', \\ (1, 0, 2) \text{ if } \nu = \lambda', \\ (1, 0, 1) \text{ otherwise} \end{cases}$$

Roughly speaking, ψ maps the i'-th column of $\mathcal{F}_n^{ab}$ to the second column of $\mathcal{F}_0$ and the λ'-th row to the second row of $\mathcal{F}_0$. The another rows and columns of $\mathcal{F}_n^{ab}$ are mapped into the first row and column of $\mathcal{F}_0$.
It is directly checked that $\psi \in \mathrm{Hom}(\mathcal{F}_n^{ab}, \mathcal{F}_0)$ and moreover $\psi(s_1) \neq \psi(s_2)$.

Thus, the existence of a homomorphism ψ proves the separation of $\mathcal{F}_n^{ab}$ by $\mathcal{F}_0$. □

According to the main results of universal algebraic geometry, the variety $\mathbf{var}(A)$, the quasi-variety $\mathbf{qvar}(A)$ and the universal closure $\mathbf{ucl}(A)$ generated by an $\mathcal{L}$-algebra A are very important in the study of equations over A (see [4, 5] for more details). The following proposition contains the statements about the universal classes generated by the c.s. semigroups $\mathcal{F}_n^{ab}$.

Proposition 4.3. *The universal classes of the semigroups $\mathcal{F}_m^{ab}, \mathcal{F}_n^{ab}$ for $n, m \geq 2$, $n > m$ satisfy the inclusions*

$$\mathbf{var}(\mathcal{F}_m^{ab}) = \mathbf{var}(\mathcal{F}_n^{ab}),\ \mathbf{qvar}(\mathcal{F}_m^{ab}) = \mathbf{qvar}(\mathcal{F}_n^{ab}),\ \mathbf{ucl}(\mathcal{F}_m^{ab}) \subset \mathbf{ucl}(\mathcal{F}_n^{ab})$$

Proof. The first equality was proved in [10]. The second one follows from Theorem 4.2 and results of [5].

Since $\mathcal{F}_m^{ab} \subseteq \mathcal{F}_n^{ab}$, the results of model theory immediately gives $\mathbf{ucl}(\mathcal{F}_m^{ab}) \subseteq \mathbf{ucl}(\mathcal{F}_n^{ab})$ Let

$$\varphi\colon\ \forall x_1 \forall x_2 \cdots \forall x_{m^2} \left(\bigwedge_{i=1}^{m^2} x_i^2 = x_i \to \bigvee_{1 \leq i < j \leq m^2} x_i = x_j \right)$$

be a universal formula of the language $\mathcal{L}$ which states that a semigroup contains at most m^2 idempotents. Clearly, φ holds in $\mathcal{F}_m^{ab}$ not in $\mathcal{F}_n^{ab}$. Thus, the definition of the universal closure immediately gives the strict inclusion $\mathbf{ucl}(\mathcal{F}_m^{ab}) \subset \mathbf{ucl}(\mathcal{F}_n^{ab})$. □

Remark 4.4. *It was proved in [7] that for free c.s. semigroups it holds $\mathbf{var}(\mathcal{F}_m) \subset \mathbf{var}(F_n)$ $(n > m)$, and it directly implies the other strict inclusions: $\mathbf{qvar}(\mathcal{F}_m) \subset \mathbf{qvar}(F_n)$, $\mathbf{ucl}(\mathcal{F}_m) \subset \mathbf{ucl}(F_n)$.*

5. Coordinate Semigroups. Equations with Constants

Let S' be a c.s. semigroup and consider the extended language $\mathcal{L}(S') = \mathcal{L} \cup \{s' \mid s' \in S'\}$. The new constant symbols correspond to all elements of S'. According to model theory, an $\mathcal{L}(S')$-term in variables $X = \{x_1, x_2, \ldots, x_n\}$ is one of the following

(1) a variable x_i;
(2) a constant symbol $s' \in S'$;
(3) a product $t(X)s(X)$ of two $\mathcal{L}(S')$-terms $t(X), s(X)$;
(4) $(t(X))^{-1}$, where $t(X)$ is an $\mathcal{L}(S')$-term.

For example, the following expressions $x(s'y)^{-1}$, $((s_1'x)^{-1}(ys_2')^{-1})^{-1}$ are $\mathcal{L}(S')$-terms. Obviously, the class of $\mathcal{L}$-terms is included into the class of all $\mathcal{L}(S')$-terms. The definitions of equations, algebraic sets and coordinate semigroups in the language $\mathcal{L}(S')$ are similar to the corresponding notions in the language $\mathcal{L}$. Let us consider the differences between coordinate algebras in the languages $\mathcal{L}$ and $\mathcal{L}(S')$.

Recall that any constant $s' \in S'$ is an $\mathcal{L}(S')$-term and $s_1' \not\sim_Y s_2'$ for any nonempty algebraic set Y defined by $\mathcal{L}(S')$-equations over S'. Therefore, the constants form a subsemigroup isomorphic to S' in any coordinate semigroup $\Gamma_{S'}(Y)$. In other words, all coordinate semigroups of algebraic sets defined by $\mathcal{L}(S')$-systems over S' are S'-semigroups, i.e. they contain a fixed subsemigroup isomorphic to S'.

Theorem 5.1. *Let $S' = (G', \mathbf{P}', \Lambda', I')$ and $|\Lambda'|, |I'| \leq \infty$. A finitely generated S'-semigroup S is the coordinate semigroup of an irreducible algebraic set defined by an $\mathcal{L}(S')$-system over S' iff S is isomorphic to $S = (G, \mathbf{P}', \Lambda', I')$, where a G'-group G is G'-discriminated by G'.*

Proof. Let us prove the "only if" part of the statement. Since $\mathcal{L}(S')$-equations includes the class of $\mathcal{L}$-equations, the semigroups S, S' should satisfy the conditions of Theorem 3.1. Therefore, there exists a subsemigroup $T' \subseteq S'$, $T' = (G', \mathbf{P}, \Lambda, I)$ and $S = (G, \mathbf{P}, \Lambda, I)$. Since S is a S'-semigroup, $\Lambda = \Lambda'$, $I = I'$, and therefore $\mathbf{P} = \mathbf{P}'$.

The subgroup $G_S = \{(1, g, 1) | g \in G\} \cong G \subseteq S$ is mapped into a subgroup G' of S' by any homomorphism $\psi \in \mathrm{Hom}_{S'}(S, S')$. Hence, the discrimination of S implies the discrimination of the group G by G'. Since the subgroup $G'_S = \{(1, g, 1) | g \in G'\} \subseteq G_S \subseteq S$ consists of constants, G' should G'-discriminate G.

The proof of the "if" part of the statement coincides with the corresponding statement of Theorem 3.1. □

Acknowledgments

The author was supported by the grant of Russian Science Foundation (project 17-11-01117).

References

[1] G. Baumslag, A. Miasnikov and V. N. Remeslennikov, Algebraic geometry over groups, *Trends in Math.*, Int. Conf. Algorithmic problems in groups and semigroups (Lincoln, NE, May 1116, 1998), Birkhauser Boston, Boston, MA (2000) 35–50.

[2] A. H. Clifford, The free completely regular semigroup on a set, *J. Algebra*, **59** (1979) 434–451.

[3] A. H. Clifford and G. B. Preston, The algebraic theory of semigroups I, Providence, *Amer. Math. Soc.*, (1961).

[4] E. Daniyarova, A. Miasnikov and V. Remeslennikov, Unification theorems in algebraic geometry, *Algebra Discrete Math.*, **1** (2008) 80–111.

[5] E. Daniyarova, A. Miasnikov and V. Remeslennikov, Algebraic geometry over algebraic structures II: Fundations, *J. Math. Sci.*, **185** (2012) 389–416.

[6] T. Harju, J. Karhumaki and M. Petrich, *Compactness of systems of equations on completely regular semigroups*, Structures in Logic and Computer Science, Lecture Notes in Comput. Sci., 1261, Springer, Berlin, 1997 268–280.

[7] P. R. Jones, Completely simple semigroups: free products, free semigroups and varieties, *Proc. Roy. Soc. Edinburgh Sect. A*, **88** (1981) 293–313.

[8] A. Myasnikov and V. Remeslennikov, Algebraic geometry over groups II: Logical foundations, *J. Algebra*, **234** (2000) 225–276.

[9] V. V. Rasin, Free completely simple semigroups, (Russian), *Matem. Zapiski (Sverdlovsk)*, (1979) 140–151 .

[10] V. V. Rasin, On the lattice of varieties of completely simple semigroups, *Semigroup Forum*, **17** (1979) 113–122

Artem N. Shevlyakov

Sobolev Institute of Mathematics Pevtsova st. 13 644099 Omsk, Russia

Omsk State Technical University, pr. Mira 11 644050 Omsk, Russia

Email: a_shevl@mail.ru