DIFFERENCE BASES IN DIHEDRAL GROUPS

TARAS BANAKH AND VOLODYMYR GAVRYLKIV

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Abstract. A subset $B$ of a group $G$ is called a difference basis of $G$ if each element $g \in G$ can be written as the difference $g = ab^{-1}$ of some elements $a, b \in B$. The smallest cardinality $|B|$ of a difference basis $B \subseteq G$ is called the difference size of $G$ and is denoted by $\Delta[G]$. The fraction $\delta[G] := \Delta[G]/\sqrt{|G|}$ is called the difference characteristic of $G$. We prove that for every $n \in \mathbb{N}$ the dihedral group $D_{2n}$ of order $2n$ has the difference characteristic $\sqrt{2} \leq \delta[D_{2n}] \leq \frac{48}{\sqrt{805}} \approx 1.983$. Moreover, if $n \geq 2 \cdot 10^{15}$, then $\delta[D_{2n}] < \frac{4}{\sqrt{8}} \approx 1.633$. Also we calculate the difference sizes and characteristics of all dihedral groups of cardinality $\leq 80$.

1. Introduction

A subset $B$ of a group $G$ is called a difference basis for a subset $A \subseteq G$ if each element $a \in A$ can be written as $a = xy^{-1}$ for some $x, y \in B$. The smallest cardinality of a difference basis for $A$ is called the difference size of $A$ and is denoted by $\Delta[A]$. For example, the set $\{0, 1, 4, 6\}$ is a difference basis for the interval $A = [-6, 6] \cap \mathbb{Z}$ witnessing that $\Delta[A] \leq 4$.

The definition of a difference basis $B$ for a set $A$ in a group $G$ implies that $|A| \leq |B|^2$ and gives a lower bound $\sqrt{|A|} \leq \Delta[A]$. The fraction

$$\delta[A] := \frac{\Delta[A]}{\sqrt{|A|}} \geq 1$$

is called the difference characteristic of $A$.

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For a real number $x$ we put

$$[x] = \min\{n \in \mathbb{Z} : n \geq x\} \text{ and } \lceil x \rceil = \max\{n \in \mathbb{Z} : n \leq x\}.$$  

The following proposition is proved in [3, 1.1].

**Proposition 1.** Let $G$ be a finite group. Then

1. $$\frac{1+\sqrt{4|G|-3}}{2} \leq \Delta[G] \leq \left\lceil \frac{|G|+1}{2} \right\rceil,$$
2. $$\Delta[G] \leq \Delta[H] \cdot \Delta[G/H] \text{ and } \partial[G] \leq \partial[H] \cdot \partial[G/H] \text{ for any normal subgroup } H \subset G;$$
3. $$\Delta[G] \leq |H| + |G/H| - 1 \text{ for any subgroup } H \subset G.$$

In [10] Kozma and Lev proved (using the classification of finite simple groups) that each finite group $G$ has difference characteristic $\bar{\partial}[G] \leq \frac{4}{\sqrt{3}} \approx 2.3094$.

In this paper we shall evaluate the difference characteristics of dihedral groups and prove that each dihedral group $D_{2n}$ has $\bar{\partial}[D_{2n}] \leq \frac{48}{\sqrt{896}} \approx 1.983$. Moreover, if $n \geq 2 \cdot 10^{15}$, then $\partial[D_{2n}] < \frac{4}{\sqrt{6}} \approx 1.633$.

We recall that the dihedral group $D_{2n}$ is the isometry group of a regular $n$-gon. The dihedral group $D_{2n}$ contains a normal cyclic subgroup of index 2. A standard model of a cyclic group of order $n$ is the multiplicative group

$$C_n = \{z \in \mathbb{C} : z^n = 1\}$$

of $n$-th roots of 1. The group $C_n$ is isomorphic to the additive group of the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Difference bases have applications in the study of structure of superextensions of groups, see [1, 3].

A subset $B$ of a group $G$ is called a basis of $G$ if each element $g \in G$ can be written as $g = ab$ for some $a, b \in B$. Bases in dihedral groups were studied in [7].

**Theorem 2.** For any numbers $n, m \in \mathbb{N}$ the dihedral group $D_{2nm}$ has the difference size

$$2\sqrt{nm} \leq \Delta[D_{2nm}] \leq \Delta[D_{2n}] \cdot \Delta[C_m]$$

and the difference characteristic $\sqrt{2} \leq \bar{\partial}[D_{2nm}] \leq \bar{\partial}[D_{2n}] \cdot \bar{\partial}[C_m]$.

**Proof.** It is well-known that the dihedral group $D_{2nm}$ contains a normal cyclic subgroup of order $nm$, which can be identified with the cyclic group $C_{nm}$. The subgroup $C_m \subset C_{nm}$ is normal in $D_{2nm}$ and the quotient group $D_{2nm}/C_m$ is isomorphic to $D_{2n}$. Applying Proposition 1(2), we obtain the upper bounds $\Delta[D_{2n}] \leq \Delta[D_{2nm}/C_m] \cdot \Delta[C_m] = \Delta[D_{2n}] \cdot \Delta[C_m]$ and $\bar{\partial}[D_{2nm}] \leq \bar{\partial}[D_{2n}] \cdot \bar{\partial}[C_m]$.

Next, we prove the lower bound $2\sqrt{nm} \leq \Delta[D_{2nm}]$. Fix any element $s \in D_{2nm} \setminus C_{nm}$ and observe that $s = s^{-1}$ and $xs^{-1} = x^{-1}$ for all $x \in C_{nm}$. Fix a difference basis $D \subset D_{2nm}$ of cardinality $|D| = \Delta[D_{2nm}]$ and write $D$ as the union $D = A \cup sB$ for some sets $A, B \subset C_{nm} \subset D_{2nm}$. We claim that $AB^{-1} = C_{nm}$. Indeed, for any $x \in C_{nm}$ we get $xs \in sC_{nm} \cap (A \cup sB)(A \cup sB)^{-1} = AB^{-1}s^{-1} \cup sBA^{-1}$ and hence

$$x \in AB^{-1}s^{-1} \cup sBA^{-1} = AB^{-1} \cup B^{-1}\ A = AB^{-1}.$$
So, $C_{nm} = AB^{-1}$ and hence $nm \leq |A| \cdot |B|$. Then $\Delta[D_{2nm}] = |A| + |B| \geq \min\{l + k : l, k \in \mathbb{N}, \ l k \geq nm\} \geq 2\sqrt{nm}$ and $\delta[D_{2nm}] = \frac{\Delta[D_{2nm}]}{\sqrt{2nm}} \geq \frac{2\sqrt{nm}}{\sqrt{2nm}} = \sqrt{2}$.

**Theorem 5.** If $\Delta[D_{2nm}] = |A| + |B| \geq \min\{l + k : l, k \in \mathbb{N}, \ l k \geq nm\} \geq 2\sqrt{nm}$ and $\delta[D_{2nm}] = \frac{\Delta[D_{2nm}]}{\sqrt{2nm}} \geq \frac{2\sqrt{nm}}{\sqrt{2nm}} = \sqrt{2}$.

**Corollary 3.** For any number $n \in \mathbb{N}$ the dihedral group $D_{2n}$ has the difference size $2\sqrt{n} \leq \Delta[D_{2n}] \leq 2 \cdot \Delta[C_n]$ and the difference characteristic $\sqrt{2} \leq \delta[D_{2n}] \leq \sqrt{2} \cdot \delta[C_n]$.

The difference sizes of finite cyclic groups were evaluated in [2] with the help of the difference sizes of the order-intervals $[1, n] \cap \mathbb{Z}$ in the additive group $\mathbb{Z}$ of integer numbers. For a natural number $n \in \mathbb{N}$ by $\Delta[n]$ we shall denote the difference size of the order-interval $[1, n] \cap \mathbb{Z}$ and by $\delta[n] := \frac{\Delta[n]}{\sqrt{n}}$ its difference characteristic. The asymptotics of the sequence $(\delta[n])_{n=1}^{\infty}$ was studied by Rédei and Rényi [11], Leech [9] and Golay [8] who eventually proved that

$$\sqrt{2 + \frac{4}{3\pi}} < \sqrt{2 + \max_{0 < \varphi < 2\pi} \frac{2\sin(\varphi)}{\varphi + \pi}} \leq \lim_{n \to \infty} \delta[n] = \inf_{n \in \mathbb{N}} \delta[n] \leq \delta[6166] = \frac{128}{\sqrt{6166}} < \delta[6] = \sqrt{\frac{8}{3}}.$$  

In [2] the difference sizes of the order-intervals $[1, n] \cap \mathbb{Z}$ were applied to give upper bounds for the difference sizes of finite cyclic groups.

**Proposition 4.** For every $n \in \mathbb{N}$ the cyclic group $C_n$ has difference size $\Delta[C_n] \leq \Delta\left[\left\lceil \frac{n-1}{2} \right\rceil \right]$, which implies that

$$\limsup_{n \to \infty} \delta[C_n] \leq \frac{1}{\sqrt{2}} \inf_{n \in \mathbb{N}} \delta[n] \leq \frac{64}{\sqrt{3083}} < \frac{2}{\sqrt{3}}.$$  

The following upper bound for the difference sizes of cyclic groups were proved in [2].

**Theorem 5.** For any $n \in \mathbb{N}$ the cyclic group $C_n$ has the difference characteristic:

1. $\delta[C_n] \leq \delta[C_4] = \frac{3}{2}$;
2. $\delta[C_n] \leq \delta[C_2] = \delta[C_8] = \sqrt{2}$ if $n \neq 4$;
3. $\delta[C_n] \leq \frac{12}{\sqrt{73}} < \sqrt{2}$ if $n \geq 9$;
4. $\delta[C_n] \leq \frac{24}{\sqrt{293}} < \frac{12}{\sqrt{73}}$ if $n \geq 9$ and $n \neq 292$;
5. $\delta[C_n] < \frac{2}{\sqrt{3}}$ if $n \geq 2 \cdot 10^{15}$.

For some special numbers $n$ we have more precise upper bounds for $\Delta[C_n]$. A number $q$ is called a prime power if $q = p^k$ for some prime number $p$ and some $k \in \mathbb{N}$.

The following theorem was derived in [2] from the classical results of Singer [13], Bose, Chowla [4], [5] and Rusza [12].

**Theorem 6.** Let $p$ be a prime number and $q$ be a prime power. Then

1. $\Delta[C_{q^2+q+1}] = q + 1$;
2. $\Delta[C_{q^2-1}] \leq q - 1 + \Delta[C_{q-1}] \leq q - 1 + \frac{3}{2}\sqrt{q-1}$;
3. $\Delta[C_{p^2-p}] \leq p - 3 + \Delta[C_p] + \Delta[C_{p-1}] \leq p - 3 + \frac{3}{2}(\sqrt{p} + \sqrt{p-1})$. 

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The following Table 1 of difference sizes and characteristics of cyclic groups $C_n$ for $n \leq 100$ is taken from [2].

**Table 1.** Difference sizes and characteristics of cyclic groups $C_n$ for $n \leq 100$.

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[http://dx.doi.org/10.22108/ijgt.2017.21612](http://dx.doi.org/10.22108/ijgt.2017.21612)
Using Theorem 6(1), we shall prove that for infinitely many numbers \( n \) the lower and upper bounds given in Theorem 2 uniquely determine the difference size \( \Delta[D_{2n}] \) of \( D_{2n} \).

**Theorem 7.** If \( n = 1 + q + q^2 \) for some prime power \( q \), then

\[
\Delta[D_{2n}] = 2 \cdot \Delta[C_n] = \left\lceil 2\sqrt{n} \right\rceil = \left\lceil \sqrt{2[D_{2n}]} \right\rceil = 2 + 2q.
\]

**Proof.** By Theorem 6(1), \( \Delta[C_n] = 1 + q \). Since

\[
2\sqrt{q^2 + q + 1} = 2\sqrt{n} \leq \Delta[D_{2n}] \leq \Delta[D_2] \cdot \Delta[C_n] = 2 \cdot \Delta[C_n] = 2 + 2q,
\]

it suffices to check that \((2 + 2q) - 2\sqrt{q^2 + q + 1} < 1\), which is equivalent to \(\sqrt{q^2 + q + 1} > q + \frac{1}{2}\) and to \(q^2 + q + 1 > q^2 + q + \frac{1}{4}\). \(\square\)

A bit weaker result holds also for the dihedral groups \( D_{8(q^2+q+1)} \).

**Proposition 8.** If \( n = 1 + q + q^2 \) for some prime power \( q \), then

\[
4q + 3 \leq \Delta[D_{8n}] \leq 4q + 4.
\]

**Proof.** By Theorem 6(1), \( \Delta[C_n] = 1 + q \). Since \( \Delta[D_8] = 4 \) (see Table 2), by Theorem 2,

\[
4\sqrt{q^2 + q + 1} = 2\sqrt{4n} \leq \Delta[D_{8n}] \leq \Delta[D_8] \cdot \Delta[C_n] = 4(1 + q).
\]

To see that \(4q + 3 \leq \Delta[D_{8n}] \leq 4q + 4\), it suffices to check that \((4 + 4q) - 4\sqrt{q^2 + q + 1} < 2\), which is equivalent to \(\sqrt{q^2 + q + 1} > q + \frac{1}{2}\) and to \(q^2 + q + 1 > q^2 + q + \frac{1}{4}\). \(\square\)

In Table 2 we present the results of computer calculation of the difference sizes and characteristics of dihedral groups of order \( \leq 80 \). In this table \( \text{lb}[D_{2n}] := \left\lceil \sqrt{4n} \right\rceil \) is the lower bound given in Theorem 2. With the boldface font we denote the numbers \( 2n \in \{14, 26, 42, 62\} \), equal to \(2(q^2 + q + 1)\) for a prime power \( q \). For these numbers we know that \( \Delta[D_{2n}] = \text{lb}[D_{2n}] = 2q + 2 \). For \( q = 2 \) and \( n = q^2 + q + 1 = 7 \) the table shows that \( \Delta[D_{56}] = \Delta[D_{8n}] = 11 = 4q + 3 \), which means that the lower bound \( 4q + 3 \) in Proposition 8 is attained.

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Table 2. Difference sizes and characteristics of dihedral groups $D_{2n}$ for $2n \leq 80$.

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<th>$\Delta[D_{2n}]$</th>
<th>$2\Delta[C_n]$</th>
<th>$\text{avg}[D_{2n}]$</th>
<th>$2n$</th>
<th>$lb[D_{2n}]$</th>
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**Theorem 9.** For any number $n \in \mathbb{N}$ the dihedral group $D_{2n}$ has the difference characteristic

$$\sqrt{2} \leq \text{avg}[D_{2n}] \leq \frac{48}{\sqrt{586}} \approx 1.983.$$  

Moreover, if $n \geq 2 \cdot 10^{15}$, then $\text{avg}[D_{2n}] < \frac{4}{\sqrt{6}} \approx 1.633$.

**Proof.** By Corollary 3, $\sqrt{2} \leq \text{avg}[D_{2n}] \leq \sqrt{2} \cdot \text{avg}[C_n]$. If $n \geq 9$ and $n \neq 292$, then $\text{avg}[C_n] \leq \frac{24}{\sqrt{293}}$ by Theorem 5(4), and hence $\text{avg}[D_{2n}] \leq \sqrt{2} \cdot \text{avg}[C_n] \leq \sqrt{2} \cdot \frac{24}{\sqrt{293}} = \frac{48}{\sqrt{586}}$. If $n = 292$, then known values $\text{avg}[C_{73}] = \frac{9}{\sqrt{73}}$ (given in Table 1), $\text{avg}[D_8] = \frac{4}{\sqrt{8}} = \sqrt{2}$ (given in Table 2) and Theorem 2 yield the upper bound

$$\text{avg}[D_{292}] = \text{avg}[D_{8,73}] \leq \text{avg}[D_8] \cdot \text{avg}[C_{73}] = \sqrt{2} \cdot \frac{9}{\sqrt{73}} < \frac{48}{\sqrt{586}}.$$  

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Analyzing the data from Table 2, one can check that $\delta[D_{2n}] \leq \frac{48}{\sqrt{586}} \approx 1.983$ for all $n \leq 8$. If $n \geq 2 \cdot 10^{15}$, then $\delta[C_n] < \frac{2}{\sqrt{n}}$ by Theorem 5(5), and hence

$$\delta[D_{2n}] \leq \sqrt{2} \cdot \delta[C_n] < \frac{4}{\sqrt{6}}.$$ 

\[ \blacksquare \]

**Question 10.** Is $\sup_{n \in \mathbb{N}} \delta[D_{2n}] = \delta[D_{22}] = \frac{8}{\sqrt{22}} \approx 1.7056$?

To answer Question 10 affirmatively, it suffices to check that $\delta[D_{2n}] \leq \frac{8}{\sqrt{22}}$ for all $n < 1212464$.

**Proposition 11.** The inequality $\delta[D_{2n}] \leq \sqrt{2} \cdot \delta[C_n] \leq \frac{8}{\sqrt{22}}$ holds for all $n \geq 1212464$.

**Proof.** It suffices to prove that $\delta[C_n] \leq \frac{4}{\sqrt{n}}$ for all $n \geq 1212464$. To derive a contradiction, assume that $\delta[C_n] > \frac{4}{\sqrt{n}}$ for some $n \geq 1212464$. Let $(q_k)_{k=1}^\infty$ be an increasing enumeration of prime powers. Let $k \in \mathbb{N}$ be the unique number such that $12q_k^2 + 14q_k + 15 < n \leq 12q_{k+1}^2 + 14q_{k+1} + 15$. By Corollary 4.9 of [2], $\Delta[C_n] \leq 4(q_k+1)$. The inequality $\delta[C_n] > \frac{4}{\sqrt{n}}$ implies

$$4(q_k+1) \geq \Delta[C_n] > \frac{4}{\sqrt{n}} \sqrt{n} \geq \frac{4}{\sqrt{n}} \sqrt{12q_k^2 + 14q_k + 16}.$$ 

By Theorem 1.9 of [6], if $q_k \geq 3275$, then $q_{k+1} \leq q_k + \frac{q_k}{2 \ln^2(q_k)}$. On the other hand, using WolframAlpha computational knowledge engine it can be shown that the inequality $1 + x + \frac{x}{2 \ln^2(x)} \leq \frac{1}{\sqrt{11}} \sqrt{12x^2 + 14x + 16}$ holds for all $x \geq 43$. This implies that $q_k < 3275$.

Analyzing the table\(^1\) of (maximal gaps between) primes, it can be shows that $11(q_{k+1}+1)^2 \leq 12q_k^2 + 14q_k + 16$ if $q_k \geq 331$. So, $q_k \leq 317$, $q_{k+1} \leq 331$ and $11 \cdot (q_{k+1}+1)^2 = 11 \cdot 332^2 = 1212464 \leq n$, which contradicts $4(q_k+1) > \frac{4}{\sqrt{n}} \sqrt{n}$.

\[ \blacksquare \]

**References**


\(^1\)See https://primes.utm.edu/notes/GapsTable.html and https://primes.utm.edu/lists/small/1000.txt


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