

SYLOW MULTIPLICITIES IN FINITE GROUPS

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ABSTRACT. Let G be a finite group and let $\mathcal{P} = P_1, \dots, P_m$ be a sequence of Sylow p_i -subgroups of G , where p_1, \dots, p_m are the distinct prime divisors of $|G|$. The Sylow multiplicity of $g \in G$ in \mathcal{P} is the number of distinct factorizations $g = g_1 \cdots g_m$ such that $g_i \in P_i$. We review properties of the solvable radical and the solvable residual of G which are formulated in terms of Sylow multiplicities, and discuss some related open questions.

1. Introduction

Sylow subgroups play a fundamental role in finite group theory, and their special properties are closely related to the fact that their orders are the maximal prime power divisors of the group's order. It is therefore natural to consider, for a general finite group, products of Sylow subgroups, one for each prime divisor of the group's order.

The following definitions and notations will be used ([9]-[11],[13]). Let G be a group and let $\pi(G) = \{p_1, \dots, p_m\}$ denote the set of all distinct prime divisors of $|G|$. For $p \in \pi(G)$, the set of all Sylow p -subgroups of G is denoted $Syl_p(G)$. With each permutation τ of $1, \dots, m$ we associate, in a natural manner, an ordering of $\pi(G)$. A *complete Sylow sequence* of G of *type* τ is a sequence of the form $\mathcal{P} = P_{\tau(1)}, \dots, P_{\tau(m)}$, where $P_j \in Syl_{p_j}(G)$. The corresponding setwise product $\Pi(\mathcal{P}) = P_{\tau(1)} \cdots P_{\tau(m)} = \{g_{\tau(1)} \cdots g_{\tau(m)} | g_i \in P_i, 1 \leq i \leq m\}$, which is a subset of G , is called a *complete Sylow product* of G of *type* τ . Whenever we wish to discuss some unspecified general ordering τ , we simplify the notation and let $\mathcal{P} = P_1, \dots, P_m$ represent a fixed but otherwise arbitrary ordering of the primes. The set of all complete Sylow sequences of G (of type τ) is denoted $CSS(G)$ ($CSS_\tau(G)$). A *factorization* of

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$g \in G$ in a complete Sylow sequence \mathcal{P} is a sequence g_1, \dots, g_m , where $g_i \in P_i$ and $g = g_1 \cdots g_m$. The multiplicity of g in \mathcal{P} , denoted $m_{\mathcal{P}}(g)$, is the number of distinct factorizations of g in \mathcal{P} .

Using this terminology we can easily reformulate a well-known nilpotency criterion: G is nilpotent if and only if it has a unique Sylow sequence for one (any) fixed ordering of $\pi(G)$. The following theorem offers a solvability criterion which is formulated in terms of Sylow multiplicities. Note that since $|G| = |P_1| \cdots |P_m|$, the condition $\Pi(\mathcal{P}) = G$ is equivalent to the condition $m_{\mathcal{P}}(g) = 1$ for every $g \in G$.

Theorem 1. *Let G be a finite group. The following three conditions are equivalent:*

- a. G is solvable.
- b. $\mathcal{P} \in CSS(G)$ and any $g \in G$, $m_{\mathcal{P}}(g) = 1$.
- c. There exists an ordering of $\pi(G)$, such that for any $\mathcal{P} \in CSS_{\tau}(G)$ and any $g \in G$, $m_{\mathcal{P}}(g) = 1$.

We briefly review the history of this result. G. Miller ([17]), and later and independently Philip Hall in ([4]), asked whether the condition $\Pi(\mathcal{P}) = G$ for every complete Sylow sequence \mathcal{P} of G implies the solvability of G (the other direction is elementary). The answer had to wait the work of John Thompson on the classification of N -groups ([18], Corollary 3), which, however, does not refer directly to Sylow sequences. Explicit proofs of Theorem 1 which are based on Thompson's results were first published in [1] (Theorem 1 - the equivalence of (a) and (b)), and later and independently in [9] (Theorem A).

What sort of questions about Sylow multiplicities can be asked when G is non-solvable? We shall say that a group G is τ -Sylow factorizable ([10],[13]) if there exists an ordering τ of $\pi(G)$ and $\mathcal{P} \in CSS_{\tau}(G)$ such that $\Pi(\mathcal{P}) = G$. Holt and Rowly ([6]) gave examples of non-solvable τ -Sylow factorizable groups, but they have also shown, via computerized calculations, that $SU(3,3)$ is not τ -Sylow factorizable for any τ . In the following we review further questions and results concerning the Sylow multiplicities $m_{\mathcal{P}}(g)$.

2. Sylow multiplicities and the solvable radical

Recall that for any finite group G the product of all subgroups of G which are both normal and solvable is a normal and solvable subgroup, called the solvable radical of G . The solvable radical of G will be denoted $R(G)$. Note that G is solvable if and only if $G = R(G)$, and so, in view of Theorem 1, it is natural to ask for a characterization of the solvable radical in terms of Sylow multiplicities. We define some subsets which are related to this question.

Definition 2. *Let G be a finite group and let τ be some fixed ordering of $\pi(G)$. Then:*

- a. $H_{\tau}(G)$ is the set of all $g \in G$ satisfying $m_{\mathcal{P}}(g) > 0$ for every $\mathcal{P} \in CSS_{\tau}(G)$, and $H(G) = \bigcap_{\text{all } \tau} H_{\tau}(G)$.
- b. $M_{\tau}(G)$ is the set of all $g \in G$ satisfying $m_{\mathcal{P}}(gx) = m_{\mathcal{P}}(x)$ for every $\mathcal{P} \in CSS_{\tau}(G)$ and $x \in G$, and $M(G) = \bigcap_{\text{all } \tau} M_{\tau}(G)$.

- c. $U_\tau(G)$ is the set of all $g \in G$ for which there exists $\mathcal{P} \in CSS_\tau(G)$, such that $m_{\mathcal{P}}(g) = 1$, and $U(G) = \bigcap_{all \tau} U_\tau(G)$.

Theorem 3 ([11] Theorem 9). *Let G be a group and let τ be some ordering of $\pi(G)$. Then:*

- a. $M_\tau(G)$ and $H_\tau(G)$ are characteristic subgroups of G .
- b. $R(G) \leq M_\tau(G) \leq H_\tau(G)$, and consequently $R(G) \leq M(G) \leq H(G)$.

In fact, it turns out (see Corollary 15 below), that $M_\tau(G) = M(G)$ for any ordering τ of $\pi(G)$. Moreover, up till now there is no example of a group G and an ordering τ which do not satisfy $R(G) = M_\tau(G) = H_\tau(G)$, and therefore the characterization of the class of groups which satisfy one or both of these equalities is an interesting open problem.

We shall say that G is τ -unique1 if $U_\tau(G) \neq \emptyset$, and that G is unique1 if $U(G) \neq \emptyset$ ¹. Note that by Lemma 15 of [11], if $U_\tau(G) \neq \emptyset$ for all τ then $U(G) \neq \emptyset$. Note that if G is τ -Sylow factorizable then it is clearly τ -unique1. Thus, one can think of $U_\tau(G) \neq \emptyset$ as a refinement of the τ -Sylow factorizability condition. It is an open problem whether every finite group G is unique1 or at least τ -unique1 for some τ . The following theorem ([11] Theorems 3,10) relates it to the previous problem.

Theorem 4. *Let G be a group. Fix some arbitrary ordering τ of $\pi(G)$.*

- a. *If all composition factors of G are τ -unique1, then $R(G) = M_\tau(G) = H_\tau(G) = H(G) = M(G)$.*
- b. *If G is τ -unique1 then $R(G) = M_\tau(G) = M(G)$.*

The problem of whether every finite group G is unique1 seems to be difficult. However, it has the advantage that it reduces to simple non-abelian groups.

Theorem 5 ([11] Theorem 4). *Fix some arbitrary ordering τ over the set of primes. The group property τ -unique1 is closed under extensions, that is, if $N \trianglelefteq G$, and G/N are both τ -unique1 then so is G . In particular, if all simple non-abelian groups are τ -unique1 then every finite group is τ -unique1.*

It is not difficult to prove that the property τ -unique1 is also inherited by normal subgroups, but at present there is no analog statement concerning quotient groups. Given that, the condition of part (a) of Theorem 4 implies the condition of part (b) but not vice versa. Interestingly, it is also easy to show that τ -Sylow factorizability is inherited by normal subgroups and quotient groups (while a general statement concerning its behavior under extensions is lacking). This is sufficient for proving:

Corollary 6. *Let G be a τ -Sylow factorizable group for some ordering τ of $\pi(G)$. Then $R(G) = M_\tau(G) = H_\tau(G) = H(G) = M(G)$.*

Proof. Since every normal subgroup and every quotient group of G is τ -Sylow factorizable we can deduce that all composition factors of G are τ -Sylow factorizable, whence, in particular, they are τ -unique1. Now the claim follows from Theorem 4. □

¹The trivial group is defined to be unique1.

Further results concerning the subgroups $H_\tau(G)$ and $H(G)$ and their relation to the solvable radical can be found in [9],[10]. Further properties of $M_\tau(G)$ and $M(G)$ and results concerning the unique property will be given below.

3. Sylow multiplicities and the solvable residual

Recall that the solvable residual of a finite group G , denoted $S(G)$, is the unique normal subgroup of G which is minimal subject to the condition that $G/S(G)$ is solvable. In order to formulate a characterization of $S(G)$ in terms of Sylow multiplicities we introduce the following quantity.

Definition 7. Let G be a group. Fix some ordering τ of $\pi(G)$. The average Sylow multiplicity of G , $m_{av}^{(G)} : G \rightarrow \mathbb{Q}_+$ (here \mathbb{Q}_+ is the set of positive rationals), is defined, for all $g \in G$, by:

$$m_{av}^{(G)}(g) \stackrel{\text{def}}{=} \frac{1}{|CSS_\tau(G)|} \sum_{\mathcal{P} \in CSS_\tau(G)} m_{\mathcal{P}}(g).$$

This quantity is of interest on its own right and will be discussed further below. In particular, it is independent of the choice of τ . The following theorem provides a characterization of $S(G)$ in terms of $m_{av}^{(G)}$. Note that $f|_N$ denotes the restriction of the function f with domain G to $N \subseteq G$.

Theorem 8 ([16] Theorem 4). Let G be a group and let $N \trianglelefteq G$. Then:

- For all $n \in N$, $m_{av}^{(N)}(n) \leq m_{av}^{(G)}(n)$.
- $S(G) \leq N$ if and only if $m_{av}^{(N)} = m_{av}^{(G)}|_N$.

Remark 9. The claim of part (a) need not be true for non-normal subgroups. A counterexample is provided by $G = A_8$ and $M < G$, $M \cong S_6$. A calculation using GAP ([3]) gives that the largest value of $m_{av}^{(M)}$ on a non-identity element is $779/675$, while the largest value of $m_{av}^{(G)}$ on a non-identity element is $48747/42875 < 779/675$.

Further results concerning the relations between the solvable residual of G and Sylow sequences of G can be found in [12].

4. The average Sylow multiplicity character

It turns out that the properly normalized average Sylow multiplicity of a finite group G (see Definition 7) is a complex character over G . For any two characters of G , χ and ψ , denote their inner product $(1/|G|) \sum_{g \in G} \chi(g) \overline{\psi(g)}$ by $\langle \chi, \psi \rangle_G$. The irreducible complex characters of G will be denoted by $\chi^{(s)}$, $1 \leq s \leq k$, where k is also the number of conjugacy classes of G , and $\chi^{(1)} = 1_G$ is the trivial character (it should be clear from the context if 1_G denotes the unit element of the group G or the trivial character of G).

Theorem 10 ([14] Theorem 5). Let $g \in G$. For all $1 \leq i \leq m$, and for any choice of $P_i \in \text{Syl}_{p_i}(G)$,

$$(4.1) \quad m_{av}^{(G)}(g) = \sum_{s=1}^k \frac{\langle 1_{P_1}, \chi^{(s)}|_{P_1} \rangle_{P_1} \cdots \langle 1_{P_m}, \chi^{(s)}|_{P_m} \rangle_{P_m}}{(\chi^{(s)}(1_G))^{m-1}} \chi^{(s)}(g).$$

Furthermore, there exists a smallest positive integer l such that $l \cdot m_{av}^{(G)}$ is a character of G .

Remark 11. Note that for any fixed $1 \leq i \leq m$, $\langle 1_{P_i}, \chi^{(s)}|_{P_i} \rangle_{P_i}$ is a non-negative integer, independent of the particular P_i chosen, and that $m_{av}^{(G)}$ is independent of the ordering of the primes τ used in Definition 7.

Theorem 8 provides a characterization of the solvable residual of G in terms of $m_{av}^{(G)}$. We can also use $m_{av}^{(G)}$ in order to prove a solvability criterion by Gallagher ([2],[7]).

Theorem 12 (Gallagher, [2]). *Let G be a group. Then G is solvable if and only if the only irreducible character χ of G satisfying $\langle 1_P, \chi|_P \rangle_P > 0$ for all Sylow subgroups P of G , is the trivial character.*

Proof. By Equation (4.1),

$$m_{av}^{(G)}(1_G) = 1 + \sum_{s=2}^k \frac{\langle 1_{P_1}, \chi^{(s)}|_{P_1} \rangle_{P_1} \cdots \langle 1_{P_m}, \chi^{(s)}|_{P_m} \rangle_{P_m}}{(\chi^{(s)}(1_G))^{m-2}}.$$

Now G is solvable if and only if $S(G) = \{1_G\}$. Since $m_{av}^{\{1_G\}}(1_G) = 1$, $S(G) = \{1_G\}$ is equivalent, by Theorem 8, to $m_{av}^{(G)}(1_G) = 1$. Thus, G is solvable if and only if $\langle 1_{P_1}, \chi^{(s)}|_{P_1} \rangle_{P_1} \cdots \langle 1_{P_m}, \chi^{(s)}|_{P_m} \rangle_{P_m} = 0$ for all $s \geq 2$, which is equivalent to Gallagher's condition. \square

Remark 13. Note that the claim of Theorem 12 is equivalent to the statement that G is solvable if and only if $m_{av}^{(G)}$ is the trivial character of G .

Next we exhibit a direct connection between the average Sylow multiplicity and the characteristic subgroups $M_\tau(G)$ (Definition 2 b). Recall ([8] Definition 2.20) that for any character χ of a group G , the kernel of χ , denoted $\ker \chi$, is the normal subgroup of G which consists of all $g \in G$ such that $\chi(g) = \chi(1_G)$. Set $\ker m_{av}^{(G)} := \ker l \cdot m_{av}^{(G)}$, where l is defined in Theorem 10.

Theorem 14 ([15] Theorem 4). *Let G be a group and let τ be an arbitrary ordering of $\pi(G)$. Then $\ker m_{av}^{(G)} = M_\tau(G)$.*

In particular we have the following corollary.

Corollary 15 ([15] Corollary 5). *Let G be any group. Then $M_\tau(G)$ is independent of τ and $M(G) = M_\tau(G) = \ker m_{av}^{(G)}$, for any ordering τ of $\pi(G)$.*

Another easy corollary is the following alternative characterization of $M_\tau(G)$.

Corollary 16. *Let G be a group and τ an ordering of $\pi(G)$. Then*

$$M_\tau(G) = \{g \in G | m_{\mathcal{P}}(g) = m_{\mathcal{P}}(1_G), \forall \mathcal{P} \in CSS_\tau(G)\}.$$

Proof. First note that if $g \in M_\tau(G)$ then $m_{\mathcal{P}}(g) = m_{\mathcal{P}}(1_G)$ for all $\mathcal{P} \in CSS_\tau(G)$ (take $x = 1_G$ in Definition 2(b)). On the other hand, if $g \in G$ satisfies $m_{\mathcal{P}}(g) = m_{\mathcal{P}}(1_G)$ for all $\mathcal{P} \in CSS_\tau(G)$, then $m_{av}^{(G)}(g) = m_{av}^{(G)}(1_G)$, and therefore $g \in \ker m_{av}^{(G)} = M_\tau(G)$. \square

Note that Theorem 14 offers a character theory characterization of $R(G)$ for every group G such that $R(G) = H(G)$, namely, $R(G)$ is the intersection of all $\ker(\chi^{(s)})$ such that

$$\langle 1_{P_1}, \chi^{(s)}|_{P_1} \rangle_{P_1} \cdots \langle 1_{P_m}, \chi^{(s)}|_{P_m} \rangle_{P_m} \neq 0$$

([8] Lemma (2.21)), and in this sense (provided that $R(G) = H(G)$ is established for all finite groups G) it could be viewed as a generalization of Gallagher's solvability criterion (Theorem 12).

We also have an easy sufficient condition for the property unique1, expressed as a condition on $m_{\text{av}}^{(G)}(1_G)$.

Proposition 17. *Let G be a group satisfying $m_{\text{av}}^{(G)}(1_G) < 2$. Then G is unique1.*

Proof. Assume to the contrary that $m_{\text{av}}^{(G)}(1_G) < 2$ but G is not unique1, and so $U(G) = \emptyset$. By Lemma 15 of [11] this implies that there exists an ordering τ such that $U_\tau(G) = \emptyset$. Thus, for all $\mathcal{P} \in \text{CSS}_\tau(G)$ and for all $g \in \Pi(\mathcal{P})$, $m_{\mathcal{P}}(g) \geq 2$. Since $1_G \in \Pi(\mathcal{P})$ for every \mathcal{P} , we get $m_{\mathcal{P}}(1_G) \geq 2$ for all $\mathcal{P} \in \text{CSS}_\tau(G)$, and hence, by Definition 7, $m_{\text{av}}^{(G)}(1_G) \geq 2$ - a contradiction. \square

The condition of Proposition 17 can be checked using GAP within the usual limitations on the group size. This way the unique1 property can be established for several "small order" simple non-abelian groups. However, for $G = A_{10}$, a GAP calculation shows that $m_{\text{av}}^{(G)}(1_G) > 2$ (A_{10} is the smallest alternating group which does not satisfy $m_{\text{av}}^{(G)}(1_G) < 2$) while another GAP calculation gives $U_\tau(G) \neq \emptyset$ for every possible τ . Thus, $m_{\text{av}}^{(G)}(1_G) < 2$ is sufficient but not necessary for G to be unique1.

For the family $G = \text{PSL}(2, q)$, where q is a prime power (G is simple non-abelian for $q \geq 4$), the structure of the character table and of the Sylow subgroups is simple enough to allow an explicit computation of $m_{\text{av}}^{(G)}(1_G)$ in terms of the maximal prime power factors of $|G|$ ([14] Theorem 12). As a corollary we obtain:

Corollary 18. *Let q be any prime power and let $G = \text{PSL}(2, q)$. Then $m_{\text{av}}^{(G)}(1_G) < 2$, and hence, by Proposition 17, $\text{PSL}(2, q)$ is unique1 for all prime powers q .*

It is clear that the method of studying the unique1 property via Proposition 17 and direct computations of $m_{\text{av}}^{(G)}(1_G)$ is of limited scope. Presumably, a new approach is required in order to make a significant advance in establishing or disproving the claim that every finite group is unique1.

5. Generalization to Profinite Groups

One of the prominent features of the theory of profinite groups is the fact that the Sylow theory of finite groups generalizes in a very natural way to the profinite setting. Hence it is natural to ask if, and to what extent, can the concepts and results presented in the previous sections be generalized to profinite groups. It seems that the answer to this question is quite satisfactory: All of the major concepts and results which were so far examined from this perspective, have natural analogs for

profinite groups. Here we will mention just a few points to give the taste of things. A full account is given in [5].

1. Given a profinite group G , the notions of a Sylow sequence of G , its product and the Sylow multiplicity of a given element in a given Sylow sequence are defined using the underlying topology.

Definition 19. Let G be a profinite group. Fix a permutation τ of the set $\text{Pr} = \{p_1, p_2, \dots\}$ of all prime numbers ordered increasingly. A complete Sylow sequence of G of type τ is a sequence $\mathcal{P} = (G_{\tau(p)})_{p \in \text{Pr}}$, where for each $p \in \text{Pr}$, G_p is a Sylow p -subgroup of G (if $p \notin \pi(G)$ then $G_p = \{1_G\}$). The product of the sequence \mathcal{P} is defined by

$$\Pi(\mathcal{P}) := \text{cl} \left(\bigcup_{i=1}^{\infty} G_{\tau(p_1)} \cdots G_{\tau(p_i)} \right),$$

where cl stands for the topological closure. An element sequence in \mathcal{P} is a sequence $(g_{\tau(p)})_{p \in \text{Pr}}$ where for each $p \in \text{Pr}$, $g_{\tau(p)} \in G_{\tau(p)}$. A factorization of g in \mathcal{P} is an element sequence $(g_{\tau(p)})_{p \in \text{Pr}}$ in \mathcal{P} such that $g = \lim_{k \rightarrow \infty} (g_{\tau(p_1)} \cdots g_{\tau(p_k)})$. We denote by $M_{\mathcal{P}}(g)$ the set of all factorizations of g in \mathcal{P} , and by $m_{\mathcal{P}}(g)$ (the Sylow multiplicity of g in \mathcal{P}) the cardinality of $M_{\mathcal{P}}(g)$.

These concepts satisfy similar relations as their finite counterparts, although some work is needed in order to show this.

Lemma 20. Let G be a profinite group, and let $\mathcal{P} = (G_{\tau(p)})_{p \in \text{Pr}}$ be a complete Sylow sequence of type τ . Let $g \in \Pi(\mathcal{P})$. Then g has a factorization in \mathcal{P} .

2. In the present context, finite group results carry over to profinite group if one replaces solvability by pro-solvability. Recall that a profinite group G is prosolvable if it is the inverse limit of an inverse system of finite solvable groups, or, equivalently, if G/N is solvable for any $N \triangleleft_O G$ (N is a normal open subgroup of G). For finite groups solvability and pro-solvability are, of course, the same. For example, we have the following generalization of Theorem 1.

Theorem 21 ([5], Theorem 1.6). Let G be a profinite group. Then the following are equivalent:

- (1) G is prosolvable.
- (2) For every permutation τ of the set of all primes and for every complete Sylow sequence \mathcal{P} of type τ it holds that $G = \Pi(\mathcal{P})$.
- (3) Fix an arbitrary permutation τ of the set of all primes. Then for every complete Sylow sequence \mathcal{P} of type τ it holds that $G = \Pi(\mathcal{P})$.

3. Any profinite group has a pro-solvable radical and the pro-solvable residual. Furthermore, one can also define the analog of $H(G)$ (see Definition 2) for a profinite G which has many of the properties of the finite object. We have:

Proposition 22 ([5], Proposition 1.7). Let G be a profinite group. If $H(G/N) = R(G/N)$ for every normal open subgroup N , then $H(G) = R(G)$.

It would be very interesting if the profinite theory will produce new insights concerning the finite $H(G) = R(G)$ question.

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