

ON METACYCLIC SUBGROUPS OF FINITE GROUPS

ADOLFO BALLESTER-BOLINCHES

Communicated by Patrizia Longobardi

ABSTRACT. The aim of this survey article is to present some structural results about of groups whose Sylow p -subgroups are metacyclic (p a prime). A complete characterisation of non-nilpotent groups whose 2-generator subgroups are metacyclic is also presented.

All groups considered here are finite.

The recovery of information about the structure of a finite group from information about its subgroups has a long history. This survey article is intended to present some contributions in this context, and it is organised around two questions concerning the influence of metacyclic Sylow subgroups and 2-generator subgroups on the group structure.

Over years there has been considerable literature studying global properties of groups which are determined by the structure or embedding of their Sylow p -subgroups, where p is a prime which is going to be fixed. Most of these results go back to Burnside's p -nilpotency criterion stating that a group is p -nilpotent, i.e. it has a normal Hall p' -subgroup provided that a Sylow p -subgroup is in the centre of its normaliser. As a consequence, a group with cyclic Sylow p -subgroups is p -nilpotent if its order is coprime to $p - 1$. This result does not remain true for metacyclic Sylow p -subgroups as the alternating group of degree 5 shows. However, if the order of a group G is coprime to $p^2 - 1$ and its Sylow p -subgroups are metacyclic, then G is p -nilpotent (see for example [11, IV, 5.10]). Therefore a natural question arises.

Question 1. *What can be said about the structure of a group with metacyclic Sylow p -subgroups?*

MSC(2010): Primary: 20D20; Secondary: 20E25.

Keywords: finite group, Sylow subgroups, metacyclic groups, 2-generator groups.

Received: 27 October 2016, Accepted: 09 March 2017.

In general we cannot say too much: consider a semidirect product of any p' -subgroup of $\text{GL}(2, p)$ with the natural module of dimension 2. However we cannot become alarmed over this remark, since we still have some interesting results.

We begin with *Sylow metacyclic groups*, or groups with all Sylow subgroups metacyclic. These groups were intensively studied by Chillag and Sonn in [7]. They proved the following structural result.

Theorem 2. *Let G be a Sylow metacyclic group. Then $G = [N]A$, where N is a normal Sylow tower subgroup of G of odd order, and either*

- G is soluble and A is a Hall subgroup of G of order $2^a 3^b$, or
- N is the largest normal subgroup of G of odd order and A is one of the following groups: M_{11} , A_7 , a metacyclic central extension A_7^+ of A_7 , $\text{PSL}(2, p^i)$, $\text{SL}(2, p^i)$, $\text{PGL}(2, p^i)$, two central extensions $\text{PGL}(2, p^i)^+$, $\text{PGL}(2, p^i)^-$ of $\text{PGL}(2, p^i)$, ($i = 1, 2$ $p^i \geq 5$), and the unique metacyclic group $\text{PGL}^*(2, p^2)$ lying between $\text{PSL}(2, p^2)$ and $\text{Aut}(\text{PSL}(2, p^2))$.

In this case, if $q \in \pi(N) \cap \pi(A)$, then the Sylow q -subgroups of G are abelian and those of A and N are cyclic.

The class of Sylow metacyclic groups is closely related to the class of \mathbb{Q} -admissible groups: a group G is said to be \mathbb{Q} -admissible if there exists a \mathbb{Q} -central division algebra containing a maximal subfield K such that $G(K/\mathbb{Q})$ is isomorphic to G . Schacher [15] proved that every \mathbb{Q} -admissible group is Sylow metacyclic. The validity of the converse is unknown at the time of writing. The following remarkable result was proved by Sonn [17].

Theorem 3. *A soluble group is Sylow metacyclic if and only if it is \mathbb{Q} -admissible*

Groups with metacyclic Sylow p -subgroups for a single p where also considered by some authors. They have a restricted structure in the p -soluble case as Monakhov and Gribovskaya proved in [14].

Theorem 4. *If G is a p -soluble group with metacyclic Sylow p -subgroups, then*

- (1) *if $p = 2$, then $G/\text{O}_{2',2}(G)$ is of odd order, or is isomorphic to Σ_3 . In particular, the 2-length of G is at most 2.*
- (2) *if $p > 2$, then the p -length of G is at most 1.*

The particular case when $p = 2$ was studied by Mazurov [13] and Camina and Gagen [6]. The latter authors showed the solubility of a group possessing a metacyclic Sylow 2-subgroup with a particular structure.

Theorem 5. *Let G be a group with a metacyclic Sylow 2-subgroup S possessing a cyclic normal subgroup N with $|S/N| > 2$. Then G is soluble.*

The results I am going to present in the sequel spring from this sources and take the study of groups with metacyclic Sylow p -subgroups further. They were proved jointly with Su and Wang in [3].

Our next theorem gives a precise description of a group with metacyclic Sylow p -subgroups provided that $G/\text{O}_{p',p}(G)$ is of odd order.

Theorem 6. *Let G be a group with metacyclic Sylow p -subgroups. Suppose that $G/O_{p'}(G)$ is of odd order, and let $G^* = G/O_p(G)$. Then G^* has a normal Sylow p -subgroup P^* , and $G^* = [P^*](H_1 \times H_2)$, where H_1 is an abelian group of exponent dividing $p - 1$, H_2 is a cyclic group with exponent dividing $p + 1$. In particular, G' is p -nilpotent.*

Let G be a group with metacyclic Sylow p -subgroups. In the following, we apply the above theorem to obtain some results on the structure of G when $|G/O_{p'}(G)|$ is coprime with $p + 1$ or $p - 1$. Recall that a group G is said to be p -supersoluble if every chief factor of G is either a cyclic group of order p or a p' -group.

Theorem 7. *Let G be a group with metacyclic Sylow p -subgroups. Assume that $|G/O_{p'}(G)|$ and $p + 1$ are coprime. Then G is p -supersoluble.*

The converse is not true: the symmetric group G of degree four has cyclic Sylow 3-subgroups, G is 3-supersoluble and $(|G/O_{3'}(G)|, 3 + 1) = 2$. However, we have:

Corollary 8. *Suppose G is a group of odd order with metacyclic Sylow p -subgroups. Then G is p -supersoluble if and only if $|G/O_{p'}(G)|$ and $p + 1$ are coprime.*

The following result, due to Berkovich [4], follows easily from the above results. It was extended by Asaad and Monakhov in [1].

Corollary 9. *Let the group $G = AB$ be the product of the subgroups A and B . If G is of odd order and the Sylow p -subgroups of A and B are cyclic, then G is p -supersoluble.*

Next, we consider what happens if G is a group with metacyclic Sylow p -subgroups such that $(|G/O_{p'}(G)|, p - 1) = 1$. In this case, G is not necessarily p -supersoluble as the following example shows:

Example 10. *Let p, q be two primes such that $2 < q$ divides $p + 1$. Let C be a cyclic group of order q and let V be an irreducible and faithful C -module over the finite field of p -elements. Then V is an elementary abelian group of order p^2 . Let $G = V \rtimes C$ be the corresponding semidirect product. Then G is a non p -supersoluble group with a metacyclic Sylow p -subgroup such that $(|G|, p - 1) = 1$.*

Theorem 11. *Assume p is odd and let G be a group with metacyclic Sylow p -subgroups. Set $G^* = G/O_{p'}(G)$. Then $(|G/O_{p'}(G)|, p - 1) = 1$ if and only if G^* satisfies the following properties:*

- (1) *A Sylow p -subgroup of G^* is normal in G^* .*
- (2) *The Hall p' -subgroups of G^* are cyclic groups of odd order dividing $p + 1$.*

The alternating group of degree 5 is an example of how the case $p = 2$ can differ from the case when p is an odd prime.

Let G be a group with metacyclic Sylow p -subgroups. Suppose that $(|G/O_{p'}(G)|, p^2 - 1) = 1$. What does G look like? The answer is contained in our next result which can be considered as an extension of [11, IV, 5.10].

Theorem 12. *Suppose that a Sylow p -subgroup of a group G is metacyclic. Then G is p -nilpotent if and only if $(|G/O_{p'}(G)|, p^2 - 1) = 1$.*

In the second part of the article, we are concerned with the influence of 2-generator subgroups on the structure of a group. The basic idea behind the results is the following: assume that a group G has 2-generator subgroups in a class \mathfrak{X} of groups which is subgroup-closed and the minimal non- \mathfrak{X} -groups are 2-generator. Then G belongs to \mathfrak{X} .

This is true for the classes of soluble [10], supersoluble [9] and nilpotent [11, Satz III.5.2] groups.

Minimal non-metacyclic p -groups have been classified by Blackburn ([5, Theorem 3.2]). These groups are all 3-generator and so the class of groups with all 2-generator subgroups metacyclic contains non-metacyclic groups. For convenience, we denote the class of finite groups with 2-generator subgroups metacyclic by \mathfrak{M} . Note that \mathfrak{M} is a subgroup and quotient closed class.

For odd primes p , the classification of p -groups in \mathfrak{M} is easy. If $G \in \mathfrak{M}$, then G can not contain a non-abelian section of order p^3 and exponent p , since such a section is not metacyclic. It then follows by [16, Lemma 2.3.3] that G is a modular group. Conversely it follows from [16, Theorem 2.3.1] that a modular p -group has all 2-generator subgroups metacyclic. Mann [12] showed that the class of monotone 2-groups coincides with the class of 2-groups in \mathfrak{M} . These groups have been classified by Crestani and Menegazzo [8] and we refer the reader to that paper for details.

In a joint paper with Cossey [2], we complete the classification of groups in \mathfrak{M} by classifying all the non-nilpotent groups in this class.

Theorem 13. *A non-nilpotent group G has 2-generator subgroups metacyclic if and only if*

- (1) *G is supersoluble and metabelian, with Sylow subgroups modular for odd primes and monotone groups for the prime 2,*
- (2) *$N = G^{\mathfrak{M}}$ (the nilpotent residual of G) is abelian (and $\neq 1$) and so $G = NK$, $N \cap K = 1$,*
- (3) *K acts on N as power automorphisms. If π is the set of primes dividing N then K_p is cyclic if $p \in \pi$ and $K_{\pi'}/C_{K_{\pi'}}(K_{\pi})$ is cyclic.*
- (4) *If $q \in \pi'$, $x \notin C_{K_q}(N)$ and $y \in C_{K_q}(N)$ then $H = \langle x, y \rangle = U\langle x \rangle$ with U cyclic, normal in H and contained in $C_H(N)$.*

Acknowledgments

This work has been supported by the grant MTM2014-54707-C3-1-P from the Ministerio de Economía y Competitividad, Spain, and FEDER, European Union. It has also been supported by a project from the National Natural Science Foundation of China (NSFC, No. 11271085), and a project of Natural Science Foundation of Guangdong Province (No. 2015A030313791).

REFERENCES

- [1] M. Asaad and V. S. Monakhov, Some Sufficient conditions for a finite group to be supersolvable, *Acta Math. Hungar.*, **135** (2012) 168–173.

- [2] A. Ballester-Bolinches and J. Cossey, Finite groups with many metacyclic subgroups, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **15** (2016) 657–662.
- [3] A. Ballester-Bolinches, N. Su and Y. Wang, On finite groups with metacyclic Sylow p -subgroups, *Publ. Math. Debrecen*, **88** (2016) 417–423.
- [4] Y. Berkovich, On solvable groups of the finite order, *Mat. Sb. (N.S.)*, **74** (1967) 75–92.
- [5] N. Blackburn, Generalizations of certain elementary theorems on p -groups, *Proc. London Math. Soc. (3)*, **11** (1961) 1–22.
- [6] A. R. Camina and T. M. Gagen, Groups with metacyclic sylow 2-subgroups, *Cand. J. Math.*, **21** (1969) 1234–1237.
- [7] D. Chillag and J. Sonn, Sylow metacyclic group and \mathbb{Q} -admissibility, *Israel J. Math.*, **40** (1981) 307–323.
- [8] E. Crestani and F. Menegazzo, On monotone 2-groups, *J. Group Theory*, **15** (2012) 359–383.
- [9] K. Doerk, Minimal nicht überauflösbare, endliche Gruppen, *Math. Z.*, **91** (1966) 198–205.
- [10] P. Flavell, Finite groups in which every two elements generate a soluble subgroup, *Invent. Math.*, **121** (1995) 279–285.
- [11] B. Huppert, *Endliche Gruppen I*, (German) Die Grundlehren der Mathematischen Wissenschaften, **134**, Springer Verlag, Berlin, Heidelberg, New York, 1967.
- [12] A. Mann, The number of generators of finite p -groups, *J. Group Theory*, **8** (2005) 317–337.
- [13] V. D. Mazurov, Finite groups with metacyclic sylow 2-subgroups, *Sibirsk. Mat. Zh.*, **8** (1967) 966–982.
- [14] V. S. Monakhov and E. E. Gribovskaya, Maximal and sylow subgroups of solvable finite groups, *Mat. Notes*, **70** (2001) 545–552.
- [15] M. Schacher, Subfields of division rings, I, *J. Algebra*, **9** (1968) 451–477.
- [16] R. Schmidt, *Subgroup lattices of groups*, De Gruyter Expositions in Mathematics, **14**, Walter de Gruyter, Berlin, 1994.
- [17] J. Sonn, \mathbb{Q} -admissibility of solvable groups, *J. Algebra*, **84** (1983) 411–419.

Adolfo Ballester-Bolinches

Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, Burjassot, València, Spain

Email: Adolfo.Ballesteruv.es