

ON EMBEDDING OF PARTIALLY COMMUTATIVE METABELIAN GROUPS TO MATRIX GROUPS

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ABSTRACT. The Magnus embedding of a free metabelian group induces the embedding of partially commutative metabelian group S_Γ in a group of matrices M_Γ . Properties and the universal theory of the group M_Γ are studied.

1. Introduction

Let Γ be a finite undirected graph with no loops and no multiple edges, $V = \{v_1, \dots, v_r\}$ is the set of the vertices of Γ , and E the set of edges of this graph.

$$F_\Gamma = \langle v_1, \dots, v_r \mid v_i v_j = v_j v_i \text{ if } (v_i, v_j) \in E \rangle$$

is a free partially commutative group.

A partially commutative metabelian group S_Γ has the same defining relations as the group F_Γ and the identity $[[x, y], [u, v]] = 1$. Properties of groups S_Γ and their universal theories were studied in [1]–[5].

The Magnus embedding (see, for example [6, 7]) enables us to consider a free metabelian group S as a subgroup of a matrix group M .

The universal theory of a free metabelian group S coincides with the universal theory of matrix group M . This fact helps Chapuis [8] to show that the universal theory of a free metabelian group is solvable. The problem on solvability of the universal theory of partially commutative metabelian group has not been solved yet. This problem is included in “The Kourovka Notebook” [9] with number 17.104.

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The Magnus embedding induces an embedding of S_Γ into a semidirect product of a free abelian group A of finite rank and an abelian normal subgroup equipped with a $\mathbb{Z}[A]$ -module structure. By M_Γ we denote this semidirect product.

We study properties of M_Γ and its universal theory. In [1] it was shown that the universal theory of M_Γ is solvable. We consider some transformations of a defining graph Γ and show that these transformations do not change the universal theory of M_Γ (Theorem 3.3). Next, we use this result to study the universal theories of so called metabelian graph products. The notion of graph product appeared in the paper [10]. After that graph products were studied in several papers (see for example [11, 12]). We consider a graph product $S(\Gamma, A_1, \dots, A_r)$ of free abelian groups A_1, \dots, A_r in the variety of metabelian groups. The group $S(\Gamma, A_1, \dots, A_r)$ is embedded to a matrix group denoted by $M(\Gamma, A_1, \dots, A_r)$. The embedding is induced by the Magnus embedding. By Theorem 3.3 the universal theories of all matrix groups $M(\Gamma, A_1, \dots, A_r)$ coincide for all free abelian groups A_1, \dots, A_r (Corollary 4).

However, given a graph Γ the universal theories of the groups S_Γ and M_Γ are different in general. For example, if L_3 is the linear graph of the length 3 then the universal theories of the partially commutative group S_{L_3} and the group of matrix M_{L_3} do not coincide. We obtain this result using the notion of centralizer dimension (Proposition 5).

Nevertheless, some common properties for the universal theories of the groups S_Γ and M_Γ are established by Theorem 4.1. This theorem states that if an equation in one variable has coefficients in S_Γ then this equation is solvable in S_Γ iff it is solvable in M_Γ .

Notice that there is the analogous result for groups of kind $F/[R, R]$, where F is a free group and R is its normal subgroup such that the ring $\mathbb{Z}[F/R]$ has no zero divisors. This result has been proved in [13].

2. Preliminaries and Notation

Let us introduce notation we use throughout this paper.

Let $V = \{v_1, \dots, v_r\}$ be the set of the vertices of a graph Γ , and E the set of edges of this graph. We consider only finite undirected graphs with no loops and no multiple edges.

Let A be the free abelian group of rank r with a basis $\{a_1, \dots, a_r\}$ and $\mathbb{Z}[A]$ the integer group ring of A . Denote by T the right free $\mathbb{Z}[A]$ -module with a basis $\{t_1, \dots, t_r\}$ and by

$$M = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$$

the matrix group.

The submodule \tilde{T}_Γ of the module T is generated by the elements

$$t_{ij} = t_i(a_j - 1) + t_j(1 - a_i)$$

such that the vertices v_i and v_j of Γ are adjacent and $i < j$. T_Γ is the factor module T/\tilde{T}_Γ .

Let S be the free metabelian group of rank r with a basis $\{s_1, \dots, s_r\}$ and R the normal subgroup generated by the commutators $[s_i, s_j] = s_i^{-1}s_j^{-1}s_is_j$ such that $(v_i, v_j) \in E$. Then the partially commutative metabelian group S_Γ is isomorphic to the factor group S/R .

The Magnus embedding μ of S to M extends the map

$$s_i \mapsto \begin{pmatrix} a_i & 0 \\ t_i & 1 \end{pmatrix}, \quad i = 1, \dots, r.$$

Define the epimorphism d of the module T to the fundamental ideal Δ of the ring $\mathbb{Z}[A]$ as follows. If $t = t_1\beta_1 + \dots + t_r\beta_r$ is in T then

$$d(t) = (a_1 - 1)\beta_1 + \dots + (a_r - 1)\beta_r.$$

In [6] it was shown that a matrix

$$\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}$$

is in the image of the group S under the Magnus embedding iff

$$d(t) = a - 1.$$

In particular, the matrix

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

is in the image of the commutant $[S, S]$ of the group S iff

$$(2.1) \quad d(t) = 0.$$

It is easy to derive that the Magnus embedding maps $[s_i, s_j]$ to

$$\begin{pmatrix} 1 & 0 \\ t_{ij} & 1 \end{pmatrix}.$$

So, the subgroup R is mapped onto the subgroup

$$\begin{pmatrix} 1 & 0 \\ \tilde{T}_\Gamma & 1 \end{pmatrix}.$$

Therefore, the Magnus embedding μ induces the embedding μ_Γ of the group S_Γ into the matrix group

$$(2.2) \quad M_\Gamma = \begin{pmatrix} A & 0 \\ T_\Gamma & 1 \end{pmatrix}.$$

A matrix

$$\begin{pmatrix} a & 0 \\ t + \tilde{T}_\Gamma & 1 \end{pmatrix} \in M_\Gamma,$$

is in the image of the group S_Γ in matrix group (2) if and only if

$$(2.3) \quad d(t) = a - 1.$$

Note that we can choose any element t in the corresponding adjacent class of T_Γ to check if condition (2.3) is satisfied.

3. Universal Equivalence of Groups M_Γ

Let v_1 be a vertex of a graph Γ , W' be the set of vertices in this graph such that these vertices are adjacent to v_1 , and $W = W' \sqcup \{v_1\}$. Denote by Γ_0 the graph obtained from Γ by adding the vertex v_0 and the edges connecting v_0 with all vertices in W .

So, the set V_0 of vertices in Γ_0 is $V \sqcup \{v_0\}$. By E_0 denote the set of edges of Γ_0 .

In [3] the vertices v_0 and v_1 were called equivalent. In this paper it was also proved that the universal theories of partially commutative metabelian groups S_Γ and S_{Γ_0} coincide (Theorem 4).

If the vertices v and w are equivalent we use notation $v \sim_\perp w$.

Let A_0 be the free abelian group with the basis $\{a_0, a_1, \dots, a_r\}$ such that A is a subgroup of A_0 and T_0 the free $\mathbb{Z}[A_0]$ -module with the basis $\{t_0, t_1, \dots, t_r\}$,

$$M_0 = \begin{pmatrix} A_0 & 0 \\ T_0 & 1 \end{pmatrix} \quad M_{\Gamma_0} = \begin{pmatrix} A_0 & 0 \\ T_{\Gamma_0} & 1 \end{pmatrix}.$$

For a natural number n consider the map ψ_n of M_0 to M that takes each matrix

$$\mathbf{m}_0 = \begin{pmatrix} a_0^{l_0} a_1^{l_1} \dots a_r^{l_r} & 0 \\ \sum_{i=0}^r t_i \alpha_i(a_0, \dots, a_r) & 1 \end{pmatrix} \in M_0$$

to the matrix

$$\mathbf{m} = \begin{pmatrix} a_1^{n l_0 + l_1} a_2^{l_2} \dots a_r^{l_r} & 0 \\ t_1 \left(\frac{a_1^n - 1}{a_1 - 1} \alpha_0(a_1^n, a_1, \dots, a_r) + \alpha_1(a_1^n, a_1, \dots, a_r) \right) + \sum_{i=2}^r t_i \alpha_i(a_1^n, a_1, \dots, a_r) & 1 \end{pmatrix}.$$

Lemma 3.1. *The map ψ_n defines a retraction of M_0 to M and $\tilde{T}_0 \psi_n = \tilde{T}$.*

Proof. For short let us use the following notation

$$\begin{aligned} \alpha_i &= \alpha_i(a_0, a_1, \dots, a_r), & \alpha'_i &= \alpha_i(a_1^n, a_1, \dots, a_r), \\ \beta_i &= \beta_i(a_0, a_1, \dots, a_r), & \beta'_i &= \beta_i(a_1^n, a_1, \dots, a_r). \end{aligned}$$

Let

$$\mathbf{n}_0 = \begin{pmatrix} a_0^{q_0} a_1^{q_1} \dots a_r^{q_r} & 0 \\ \sum_{i=1}^r t_i \beta_i & 1 \end{pmatrix} \in M_0.$$

Then

$$\begin{aligned} \mathbf{m}_0 \psi_n &= \begin{pmatrix} a_1^{n l_0 + l_1} a_2^{l_2} \dots a_r^{l_r} & 0 \\ t_1 \left(\frac{a_1^n - 1}{a_1 - 1} \alpha'_0 + \alpha'_1 \right) + \sum_{i=2}^r t_i \alpha'_i & 1 \end{pmatrix}, \\ \mathbf{n}_0 \psi_n &= \begin{pmatrix} a_1^{n q_0 + q_1} a_2^{q_2} \dots a_r^{q_r} & 0 \\ t_1 \left(\frac{a_1^n - 1}{a_1 - 1} \beta'_0 + \beta'_1 \right) + \sum_{i=2}^r t_i \beta'_i & 1 \end{pmatrix}, \\ \mathbf{m}_0 \mathbf{n}_0 &= \begin{pmatrix} a_0^{l_0 + q_0} a_1^{l_1 + q_1} \dots a_r^{l_r + q_r} & 0 \\ \sum_{i=0}^r t_i (\alpha_i a_0^{q_0} \dots a_r^{q_r} + \beta_i) & 1 \end{pmatrix}. \end{aligned}$$

Applying ψ_n to the last matrix we obtain

$$(3.1) \quad (\mathbf{m}_0 \mathbf{n}_0) \psi_n = \begin{pmatrix} a_1^{n(l_0 + q_0) + l_1 + q_1} a_2^{l_2 + q_2} \dots a_r^{l_r + q_r} & 0 \\ \tau & 1 \end{pmatrix},$$

where

$$\tau = t_1 \left(\frac{a_1^n - 1}{a_1 - 1} (\alpha'_0 a_1^{nq_0+q_1} a_2^{q_2} \cdots a_r^{q_r} + \beta'_0) + \alpha'_1 a_1^{nq_0+q_1} a_2^{q_2} \cdots a_r^{q_r} + \beta'_1 \right) + \sum_{i=2}^r t_i (\alpha'_i a_1^{nq_0+q_1} a_2^{q_2} \cdots a_r^{q_r} + \beta'_i).$$

Compute the $(2, 1)$ -entry of the matrix $(\mathbf{m}_0 \psi_n)(\mathbf{n}_0 \psi_n)$. It coincides with τ . Comparing the matrices $(\mathbf{m}_0 \mathbf{n}_0) \psi_n$ and $(\mathbf{m}_0 \psi_n)(\mathbf{n}_0 \psi_n)$ we see that they are equal.

Therefore ψ_n is a homomorphism acting identically on M . So, ψ_n is a retraction.

Let us show that $\tilde{T}_0 \psi_n = \tilde{T}$.

If $i \neq 0$ then $t_{ij} \psi_n = t_{ij}$.

Consider the images of the elements t_{0j} under ψ_n . We obtain

$$t_{01} \psi_n = (t_0(a_1 - 1) + t_1(1 - a_0)) \psi_n = t_1 \left(\frac{a_1^n - 1}{a_1 - 1} (a_1 - 1) - a_1^n + 1 \right) = 0.$$

If $j \neq 1$ then we have

$$t_{0j} \psi_n = (t_0(1 - a_j) + t_j(a_0 - 1)) \psi_n = t_1 \frac{a_1^n - 1}{a_1 - 1} (1 - a_j) + t_j(a_1^n - 1) = \frac{a_1^n - 1}{1 - a_1} t_{1j}.$$

But if $(v_0, v_j) \in E_0$ then $(v_1, v_j) \in E$. Consequently $t_{0j} \psi_n$ is in \tilde{T} . This completes the proof of the lemma. □

We identify S_Γ and S_{Γ_0} with their images in M_Γ and M_{Γ_0} respectively. By Lemma 3.1 the homomorphism ψ_n induces the homomorphism φ_n of M_{Γ_0} onto M_Γ .

Lemma 3.2. *The homomorphism φ_n has the following properties*

1. φ_n is a retraction;
2. φ_n maps S_{Γ_0} onto S_Γ ;
3. For any element $1 \neq g \in M_{\Gamma_0}$ there exists n_0 such that $g \varphi_n \neq 1$ whenever $n \geq n_0$.

Proof. The first property follows from the definition of ψ_n .

Let us prove the second property. Let S_0 be the free metabelian group with the basis $\{s_0, s_1, \dots, s_r\}$ and S its subgroup generated by s_1, \dots, s_r . Take all commutators $[s_i, s_j]$ such that $(v_i, v_j) \in E_0$ and $0 \leq i < j \leq r$. Denote by R_0 the normal closure of these commutators in S_0 .

We have

$$\begin{aligned} s_0 R_0 &= \begin{pmatrix} a_0 & 0 \\ t_0 + \tilde{T}_0 & 1 \end{pmatrix} \xrightarrow{\varphi_n} \begin{pmatrix} a_1^n & 0 \\ \frac{a_1^n - 1}{a_1 - 1} t_1 + \tilde{T} & 1 \end{pmatrix} = \\ &= \begin{pmatrix} a_1 & 0 \\ t_1 + \tilde{T} & 1 \end{pmatrix}^n = (s_1 R)^n. \end{aligned}$$

Clearly, $(s_i R_0) \varphi_n = s_i R$ for $1 \leq i \leq r$. Therefore, $S_{\Gamma_0} \psi_n = S_\Gamma$.

We are left to prove the third property.

Let

$$1 \neq g = \begin{pmatrix} a & 0 \\ t_0 \alpha_0 + \cdots + t_r \alpha_r + \tilde{T}_0 & 1 \end{pmatrix}, \quad a \in A_0, \quad \alpha_i \in \mathbb{Z}[A_0].$$

Case 1: $a \neq 1$.

If a does not depend on a_0 then any number can be chosen for n .

Let a depend on a_0 , i.e. $a = a_0^{l_0} \cdots a_r^{l_r}$, $l_0 \neq 0$. Choose n_0 such that $l_0 n_0 + l_1 > 0$ for $l_0 > 0$ and $l_0 n_0 + l_1 < 0$ for $l_0 < 0$. Obviously, for all $n \geq n_0$ we get $g\varphi_n \neq 1$.

Case 2a: $a = 1$, $\sum_{i=0}^r \alpha_i(a_i - 1) = 0$.

In this case the matrix g is in the commutant of S_{Γ_0} . The homomorphism φ_n induces the homomorphism $\bar{\varphi}_n$ of S_{Γ_0} onto S_{Γ} and for φ_n we have

$$s_0 R_0 \mapsto s_1^n R, \quad s_i R_0 \mapsto s_i R, \quad i = 1, \dots, r.$$

In [3], Theorem 4, it was shown that there exists n_0 such that for any $n \geq n_0$ the image of $g \in [S_{\Gamma_0}, S_{\Gamma_0}]$ is not equal to the unit. This value of n_0 satisfies the condition of the lemma.

Case 2b: $a = 1$, $\sum_{i=0}^r \alpha_i(a_i - 1) \neq 0$.

Let $\alpha = \sum_{i=0}^r \alpha_i(a_i - 1)$. Define a homomorphism $\chi_n : \mathbb{Z}[A_0] \rightarrow \mathbb{Z}[A]$ on the basis of A_0 as follows.

$$\chi_n = \{a_0 \mapsto a_1^n, a_1 \mapsto a_1, \dots, a_r \mapsto a_r\}.$$

Choose n_0 in such a way that the image of α under χ_n is non-zero. Show that for any $n \geq n_0$ the element $g\varphi_n$ is not equal to the unit.

We have

$$g\varphi_n = \begin{pmatrix} 1 & 0 \\ t_1 \left(\frac{a_1^n - 1}{a_1 - 1} \alpha'_0 + \alpha'_1 \right) + t_2 \alpha'_2 + \dots + t_r \alpha'_r & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} (a_1 - 1) \left(\frac{a_1^n - 1}{a_1 - 1} \alpha'_0 + \alpha'_1 \right) + (a_2 - 1) \alpha'_2 + \dots + (a_r - 1) \alpha'_r = \\ (a_1^n - 1) \alpha'_0 + (a_1 - 1) \alpha'_1 + \dots + (a_r - 1) \alpha'_r = \alpha \chi_n \neq 0. \end{aligned}$$

The lemma is proved completely. □

A group G is *discriminated* by a group H if for any set of non-unit elements $\{g_1, \dots, g_n\}$ in G there exists a homomorphism $\varphi : G \rightarrow H$ such that $\varphi(g_i) \neq 1$ for all $i = 1, \dots, n$.

It is well known that if G is discriminated by H and H is discriminated by G then the universal theories of G and H coincide.

Theorem 3.3. *Let v_0 and v_1 be equivalent $v_0 \sim_{\perp} v_1$ in a graph Γ_0 and let the graph Γ be obtained from a graph Γ_0 by deleting a vertex v_0 and all edges incident to v_0 . Then the universal theories of M_{Γ} and M_{Γ_0} coincide.*

Proof. It is clear that the group M_{Γ} is embedded in the group M_{Γ_0} . On the other hand, Lemma 3.2 implies that M_{Γ_0} is discriminated by M_{Γ} . So, the assertion follows. □

Let Γ be a graph with the set of vertices $V = \{v_1, \dots, v_r\}$ and the set of edges E . Consider a family of non-trivial groups $\{G_v \mid v \in V\}$. The *graph product* of these groups is the factor group of the free product $\prod_{v \in V}^* G_v$ by the normal subgroup generated by commutants $[G_v, G_w]$ such that $(v, w) \in E$.

Let us define the notion of metabelian graph product. Actually we need to change the free product of groups by the metabelian product. We are going to use the definition of metabelian product for free abelian groups.

Let $A_i = A_{v_i}$ be a free abelian group, $i = 1, \dots, r$.

Metabelian product $\mathfrak{A}^2 \amalg A_i$ is the factor group of the free product $F = \amalg^* A_i$ by the second commutant

$$F^{(2)} = [[F, F], [F, F]].$$

Metabelian graph product $S(\Gamma, A_1, \dots, A_r)$ is the factor of $\mathfrak{A}^2 \amalg A_i$ by the normal subgroup generated by the commutants $[A_i, A_j]$, such that $(v_i, v_j) \in E$.

Metabelian graph product of free abelian groups of finite ranks can be obtained as follows. Let A_i be the free group of rank $ri \geq 2$. For $i = 1, \dots, r$ we add the vertices $v_{i_2}, \dots, v_{i_{r_i}}$ to the set V of vertices of the graph Γ . We connect all vertices $v_i, v_{i_2}, \dots, v_{i_{r_i}}$ pairwise. In addition, we connect all vertexes $v_{i_2}, \dots, v_{i_{r_i}}$ with v whenever $(v_i, v) \in E$. Let Δ be the obtained graph. It is clear that S_Δ and $S(\Gamma, A_1, \dots, A_r)$ are isomorphic.

Denote M_Δ by $M(\Gamma, A_1, \dots, A_r)$.

Since $v_{i_2}, \dots, v_{i_{r_i}}$ are equivalent to v_i Theorem 3.3 implies the following corollary.

Corollary 3.4. *Let $A_i, i = 1, \dots, r$, be free abelian groups of finite ranks and Γ a graph. Then the universal theories of M_Γ and $M(\Gamma, A_1, \dots, A_r)$ coincide.*

Let Γ be a totally disconnected graph. Then $S_\Gamma \simeq S$ is the free metabelian group of rank r and $M_\Gamma \simeq AwrB$ is a wreath product of two abelian groups of rank r . In [8] it was shown that the universal theories of the groups S and $AwrB$ coincide.

But there exists a graph Γ such that the universal theories of S_Γ and M_Γ differ.

Proposition 3.5. *Let $\Gamma = L_3$ be the linear graph on three vertices. Then the universal theories of S_Γ and M_Γ do not coincide.*

Proof. Recall that centralizer dimension $CdimG$ of a group G is equal to n if there exist subsets

$$A_1 \subset A_2 \subset \dots \subset A_n,$$

in G such that their centralizers

$$C(A_1) > C(A_2) \dots > C(A_n)$$

are strictly decreasing and n is the largest number such that this property holds for n .

If there is no largest n then set $Cdim(G) = \infty$.

Notice that coincidence of universal (equivalently, existential) theories of two groups implies coincidence of their centralizer dimensions.

First, let us find centralizer dimension of the group S_Γ which is isomorphic to the direct product of the free metabelian group $S_2\langle x_1, x_3 \rangle$ of rank 2 and the infinite cyclic group $\langle x_2 \rangle$. In [14], the following formula for centralizer dimension of a direct product of two groups was proved

$$Cdim(S_2 \times \langle x \rangle) = Cdim(S_2) + Cdim(\langle x \rangle) - 1.$$

So, it is easy to see that $Cdim(S_\Gamma) = 3$.

Now let us find $Cdim(M_\Gamma)$. Let

$$\begin{aligned} \hat{t}_{13} &= \begin{pmatrix} 1 & 0 \\ t_1(a_3 - 1) + t_3(1 - a_1) + \tilde{T}_\Gamma & 1 \end{pmatrix}, \\ \hat{t}_1 &= \begin{pmatrix} 1 & 0 \\ t_1 + \tilde{T}_\Gamma & 1 \end{pmatrix}, \quad \hat{t}_2 = \begin{pmatrix} 1 & 0 \\ t_2 + \tilde{T}_\Gamma & 1 \end{pmatrix} \\ \hat{a}_1 &= \begin{pmatrix} a_1 & 0 \\ \tilde{T}_\Gamma & 1 \end{pmatrix}, \quad \hat{a}_2 = \begin{pmatrix} a_2 & 0 \\ \tilde{T}_\Gamma & 1 \end{pmatrix}. \end{aligned}$$

We obtain the chain of centralizers

$$M_\Gamma \stackrel{\hat{a}_1}{>} C(\hat{t}_{13}) \stackrel{\hat{a}_2}{>} C(\hat{t}_{13}, \hat{t}_2) \stackrel{\hat{t}_1}{>} C(\hat{t}_{13}, \hat{t}_2, \hat{a}_2).$$

In detail:

1. $[\hat{a}_1, \hat{t}_{13}] = [x_1, [x_1, x_3]] \neq 1$ in S_2 .
2. $[\hat{a}_2, \hat{t}_{13}] = [x_2, [x_1, x_3]] = 1$. If $[\hat{a}_2, \hat{t}_2] = 1$ then $t_2(a_2 - 1)$ is in the submodule \tilde{T}_Γ of the free module T generated by the elements

$$(a_1 - 1)t_2 + (1 - a_2)t_1, \quad (a_3 - 1)t_2 + (1 - a_2)t_3.$$

But this is impossible.

3. For the same reason $\hat{t}_1 \in C(\hat{t}_{13}, \hat{t}_2) \setminus C(\hat{t}_{13}, \hat{t}_2, \hat{a}_2)$.

Therefore $Cdim(M_\Gamma) \geq 4$. This concludes the proof. □

4. Equations in one unknown

Theorem 4.1. *An equation*

$$(4.1) \quad g_1 x^{m_1} \cdots g_l x^{m_l} = 1, \quad g_i \in S_\Gamma,$$

is solvable in S_Γ iff it is solvable in M_Γ .

Proof. Let

$$\mu_\Gamma : g_j \mapsto \hat{g}_j = \begin{pmatrix} b_j & 0 \\ \tau_j + \tilde{T}_\Gamma & 1 \end{pmatrix}, \quad j = 1, \dots, l, \quad b_j \in A, \quad \tau_j \in T.$$

Suppose that equation (4.1) is solvable in M_Γ and

$$\hat{x} = \begin{pmatrix} x & 0 \\ \tau & 1 \end{pmatrix}$$

is its solution.

One can compute that the left-hand side of the equation is equal to

$$\begin{pmatrix} b_1 \cdots b_l x^{m_1 + \cdots + m_l} & 0 \\ \gamma & 1 \end{pmatrix},$$

where

$$\begin{aligned} \gamma &= \tau_l x^{m_l} + \sum_{j=1}^{l-1} \tau_j b_{j+1} \cdots b_l x^{m_j + \cdots + m_l} + \\ &\tau \left(\frac{x^{m_l} - 1}{x - 1} + \sum_{j=1}^{l-1} b_{j+1} \cdots b_l \frac{x^{m_j} - 1}{x - 1} x^{m_{j+1} + \cdots + m_l} \right) + \tilde{T}_\Gamma, \end{aligned}$$

and $\frac{x^m - 1}{x - 1} = m$ for $x = 1$.

Equation (4.1) is solvable in M_Γ iff the system of equations

$$(4.2) \quad b_1 \cdots b_l x^{m_1 + \cdots + m_l} = 1 \quad \bigwedge \quad \gamma = 0$$

is solvable with respect to $\tau \in T$ and $x \in A$.

Let us use the following notation

$$B = \frac{x^{m_l} - 1}{x - 1} + \sum_{j=1}^{l-1} b_{j+1} \cdots b_l \frac{x^{m_j} - 1}{x - 1} x^{m_{j+1} + \cdots + m_l}.$$

Since d is a module homomorphism and $d(\tau_j) = b_j - 1$ we have

$$d(\gamma) = (b_l - 1)x^{m_l} + \sum_{j=1}^{l-1} (b_j - 1)b_{j+1} \cdots b_l x^{m_j + \cdots + m_l} + d(\tau)B.$$

Since (x, τ) is a solution of (4.2), we obtain

$$B(x - 1) + (b_l - 1)x^{m_l} + \sum_{j=1}^{l-1} (b_j - 1)b_{j+1} \cdots b_l x^{m_j + \cdots + m_l} = 0.$$

We obviously have

$$Bd(\tau) + (b_l - 1)x^{m_l} + \sum_{j=1}^{l-1} (b_j - 1)b_{j+1} \cdots b_l x^{m_j + \cdots + m_l} = d(\gamma) = 0.$$

Consequently $B(x - 1) = Bd(\tau)$.

If $B = 0$ then system (4.2) does not depend on τ . Therefore for any $y \in T_\Gamma$ the matrix

$$(4.3) \quad \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}$$

is a solution of (4.2). Evidently, for $x \in A$ one can find y such that the matrix (4.3) is in S_Γ .

If $B \neq 0$ then $d(\tau) = x - 1$. This means that the matrix \hat{x} is S_Γ . This completes the proof. □

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REFERENCES

- [1] Ch. K. Gupta and E. I. Timoshenko, Partially Commutative Metabelian Groups: Centralizers and elementary Equation, *Algebra Logic*, **48** (2009) 173–192.
- [2] E. I. Timoshenko, Universal Equivalence of Partially Commutative Metabelian Groups, *Algebra Logic*, **49** (2010) 177–196.
- [3] Ch. K. Gupta and E. I. Timoshenko, On Universal Theories of Partially Commutative Metabelian Groups, *Algebra Logic*, **50** (2011) 1–16.
- [4] E. I. Timoshenko, A Mal'tsev Basis for a Partially Commutative Nilpotent Metabelian Group, *Algebra Logic*, **50** (2011) 647–658.
- [5] Ch. K. Gupta and E. I. Timoshenko, Properties and Universal Theories of Partially Commutative Metabelian Nilpotent Groups, *Algebra Logic*, **51** (2012) 285–305.
- [6] V. N. Remeslennikov and V. G. Sokolov, Some Properties of the Magnus Embedding, (Russian), *Algebra Logic*, **9** (1970) 342–349.
- [7] E. I. Timoshenko, *Endomorphisms and Universal Theories of Solvable Groups*, Novosibirsk: NSTU publishers 2013 pp. 327
- [8] O. Chapuis, Universal Theory of Certain Solvable Groups and Bounded Ore Group Rings, *J. Algebra*, **176** (1995) 368–391.
- [9] Unsolved Problems in Group Theory, The Kourovka Notebook, issue 17, 2011.
- [10] E. R. Green, *Graph products of groups*, PhD Thesis of Newcastleupon-Tyne, 2006.
- [11] L. J. Corredor and M. A. Gutierrez, A generating set for the automorphism group of a graph product of abelian groups, *Internat. J. Algebra Comput.*, **22** (2012) pp. 21, [arXiv:0911.0576v1\[math.GR\]](https://arxiv.org/abs/0911.0576v1), 2009.
- [12] R. Charney, K. Ruane, N. Stambaugh and A. Vijayan, The automorphisms group of a graph product with no SIL, *Illinois J. Math.*, **54** (2010) 249–262, [arXiv:0910.4886](https://arxiv.org/abs/0910.4886).
- [13] E. I. Timoshenko, Metabelian Groups with one Defining Relation, and the Magnus Embedding, *Math. Notes*, **57** (1995) 414–420.
- [14] A. Myasnikov and P. Shumyatsky, Discriminating groups and c -dimension, *J. Group Theory*, **7** (2004) 135–142.

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