

## THE CONJUGACY CLASS RANKS OF $M_{24}$

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Dedicated to Professor Jamshid Moori on the occasion of his seventieth birthday

Communicated by Robert Turner Curtis

ABSTRACT.  $M_{24}$  is the largest Mathieu sporadic simple group of order  $244823040 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$  and contains all the other Mathieu sporadic simple groups as subgroups. The object in this paper is to study the ranks of  $M_{24}$  with respect to the conjugacy classes of all its nonidentity elements.

### 1. Introduction

$M_{24}$  is the largest Mathieu sporadic simple group of order  $244823040 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$  and contains all the other Mathieu sporadic simple groups as subgroups. It is a 5-transitive permutation group on a set of 24 points such that:

- (i) the stabilizer of a point is  $M_{23}$  which is 4-transitive
- (ii) the stabilizer of two points is  $M_{22}$  which is 3-transitive
- (iii) the stabilizer of a dodecad is  $M_{12}$  which is 5-transitive
- (iv) the stabilizer of a dodecad and a point is  $M_{11}$  which is 4-transitive

$M_{24}$  has a trivial Schur multiplier, a trivial outer automorphism group and it is the automorphism group of the Steiner system of type  $S(5,8,24)$  which is used to describe the Leech lattice on which  $M_{24}$  acts.  $M_{24}$  has nine conjugacy classes of maximal subgroups which are listed in [4], its ordinary character table is found in [4], its blocks and its Brauer character tables corresponding to the various primes dividing its order are found in [9] and [10] respectively and its complete prime spectrum is given by 2, 3, 5, 7, 11, 23.

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In [13] we studied some of the triple generations of  $M_{24}$ . In the present paper, we shall study the ranks of  $M_{24}$  with respect to the conjugacy classes of all its nonidentity elements. The computations were carried out using the computer algebra system GAP [8] which is running on a LINUX machine in the Department of Mathematical Sciences at the University of South Africa.

Throughout, our groups  $G$  are finite, a conjugacy class is of elements of  $G$  and  $p, q, r$  denote primes that divide the order of  $G$  unless otherwise specified to the contrary. Reference to the various maximal subgroups of  $M_{24}$  should be understood to mean up to isomorphism.

## 2. Preliminaries

For a finite group  $G$  and  $nX$  a conjugacy class of nonidentity elements of  $G$ , we define  $\text{rank}(G : nX)$  to be the minimum number of elements of  $G$  in  $nX$  that generate  $G$ . We call  $\text{rank}(G : nX)$  the **RANK** of  $G$  with respect to the conjugacy class  $nX$ . If  $C_1, C_2, \dots, C_n$  are conjugacy classes of elements of  $G$  and  $g_n \in C_n$  is a fixed representative, then  $\xi_G(C_1, C_2, \dots, C_n)$  denotes the number of distinct tuples  $(g_1, g_2, \dots, g_{n-1})$ , where  $g_i \in C_i$  for  $1 \leq i \leq n-1$  such that  $g_1 g_2 \dots g_{n-1} = g_n$ . This nonnegative integer is known as the **STRUCTURE CONSTANT** of the group algebra  $\mathbb{C}G$  and can be computed from the ordinary character table of  $G$  using the formula

$$\xi_G(C_1, C_2, \dots, C_n) = \frac{|C_1||C_2|\cdots|C_{n-1}|}{|G|} \sum_{j=1}^k \frac{\chi_j(g_1)\chi_j(g_2)\cdots\chi_j(g_{n-1})\overline{\chi_j(g_n)}}{[\chi_j(1_G)]^{n-2}}$$

where  $\chi_1, \chi_2, \dots, \chi_k \in \text{Irr}(G)$ .

Suppose that  $\xi_G^*(C_1, C_2, \dots, C_n)$  denotes the number of distinct tuples  $(g_1, g_2, \dots, g_{n-1})$  with  $g_i \in C_i$  for  $1 \leq i \leq n-1$  for which  $g_1 g_2 \dots g_{n-1} = g_n$  such that  $G = \langle g_1, g_2, \dots, g_{n-1} \rangle$ . If we get that  $\xi_G^*(C_1, C_2, \dots, C_n) > 0$ , then we say that  $G$  is  $(C_1, C_2, \dots, C_n)$ -generated. If  $H \leq G$  containing  $g_n \in C_n$  and  $B$  is a conjugacy class of elements of  $H$  such that  $g_n \in B$ , then  $\sum_H(C_1, C_2, \dots, C_{n-1}, B)$  denotes the number of distinct tuples  $(g_1, g_2, \dots, g_{n-1})$  with  $g_i \in C_i$  for  $1 \leq i \leq n-1$  for which  $g_1 g_2 \dots g_{n-1} = g_n$  such that  $\langle g_1, g_2, \dots, g_{n-1} \rangle \leq H$ . We observe that  $H \cap C_i$  for  $1 \leq i \leq n-1$  decomposes into a disjoint union of conjugacy classes of  $H$  and  $\sum_H(C_1, C_2, \dots, C_{n-1}, B)$  is obtained by summing over all such conjugacy classes of maximal subgroups  $H$  of  $G$ .

**Theorem 2.1.** [1],[12] *Let  $G$  be a finite simple group such that  $G$  is  $(lX, mY, nZ)$ -generated. Then  $G$  is  $(\underbrace{(lX, lX, \dots, lX)}_{m\text{-times}}, (nZ)^m)$ -generated.*

*Proof.* This is [1, Lemma 2] and [12, Theorem 5]. □

**Corollary 2.2.** [5] [7] *Let  $G$  be a simple group such that  $G$  is  $(2X, sY, tZ)$ -generated. Then  $G$  is  $(sY, sY, (tZ)^2)$ -generated.*

*Proof.* This is [5, Lemma 2]. □

**Theorem 2.3.** *Let  $G$  be a finite group and  $H$  a subgroup of  $G$  containing an element  $x$  such that  $(o(x), [N_G(H) : H]) = 1$ . Then the number  $h$  of conjugates of  $H$  containing  $x$  is given by  $\chi_H(x)$ , where*

$\chi_H$  is the permutation character of  $G$  acting on the conjugates of  $H$ . In particular, we obtain that

$$h = \sum_{i=1}^m \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|}$$

where  $x_1, x_2, \dots, x_m$  are representatives of the  $N_G(H)$ -conjugacy classes that fuse into the  $G$ -class  $[x]_G$ .

*Proof.* cf. [2],[6] □

The following result can be very useful in proving nongeneration.

**Lemma 2.4.** *Let  $G$  be a finite centerless group,  $lX, mY, nZ$  be conjugacy classes of  $G$  for which  $\xi_G^*(lX, mY, nZ) < |C_G(z)|$  for  $z \in nZ$ . Then  $\xi_G^*(lX, mY, nZ) = 0$  so that  $G$  is NOT  $(lX, mY, nZ)$ -generated.*

*Proof.* cf. [2],[5] □

### 3. The ranks of $M_{24}$

We study here the ranks of  $M_{24}$  with respect to the various conjugacy classes of all its nonidentity elements.

**Lemma 3.1.** [3] *Let  $G$  be a finite simple group such that  $G$  is  $(lX, mY, nZ)$ -generated. Then  $\text{rank}(G : lX) \leq m$ .*

*Proof.* Since  $G$  is  $(lX, mY, nZ)$ -generated, by Theorem 2.1 above, we get that  $G$  is generated by  $m$  elements from the class  $lX$ . Hence the result follows thus completing the proof. □

**Proposition 3.2.**  $\text{rank}(M_{24} : 2A) = 3 = \text{rank}(M_{24} : 2B)$

*Proof.* We have by [13] that  $M_{24}$  is  $(2A, 3B, 23A)$  and  $(2B, 3A, 23A)$ -generated. Thus we must have by Lemma 3.1 above that  $\text{rank}(M_{24} : 2A) \leq 3$  and  $\text{rank}(M_{24} : 2B) \leq 3$ . By Theorem 2.1, we have that  $M_{24}$  is  $(2A, 2A, 2A, (23A)^3)$  and  $(2B, 2B, 2B, (23A)^3)$ -generated. However by [8] we have that  $(23A)^3 = 23A$  so that  $M_{24}$  becomes  $(2A, 2A, 2A, 23A)$  and  $(2B, 2B, 2B, 23A)$ -generated. Also by the proof of [1, Lemma 3.1] and by [11, Proposition 1], the result follows and the proof is complete. □

**Proposition 3.3.**  $\text{rank}(M_{24} : 3A) = 2 = \text{rank}(M_{24} : 3B)$

*Proof.* According to [13], we have that  $M_{24}$  is  $(2A, 3B, 23A)$  and  $(2B, 3A, 23B)$ -generated. Only the maximal subgroups  $M_{23}$  and  $L_2(23)$  of  $M_{24}$  have elements of order 23. We obtain that  $\xi_{M_{24}}(3A, 3A, 23B) = 161$  and  $\xi_{M_{24}}(3B, 3B, 23A) = 552$ . We have that  $M_{23}$  meets the  $3A$  and not the  $3B$  class whereas  $L_2(23)$  meets the  $3B$  and not the  $3A$  class of  $M_{24}$ . We thus obtain that  $\xi_{M_{23}}(3a, 3a, 23b) = 138$  and  $\xi_{M_{L_2(23)}}(3a, 3a, 23a) = 23$ . By Theorem 2.3 above, the classes  $23b$  of  $M_{23}$  and  $23a$  of  $L_2(23)$  are contained in one conjugate each of  $M_{23}$  and  $L_2(23)$  respectively. Thus we obtain that  $\xi_{M_{24}}^*(3A, 3A, 23B) = 161 - 138 = 23 = |C_{M_{24}}(23B)|$  and  $\xi_{M_{24}}^*(3B, 3B, 23A) = 552 - 23 = 529 > |C_{M_{24}}(23A)| = 23$ . Thus we have that  $M_{24}$  is  $(3B, 3B, 23A)$  and  $(3A, 3A, 23B)$ -generated. Hence the result follows. □

**Proposition 3.4.**  $\text{rank}(M_{24} : 4A) = 2 = \text{rank}(M_{24} : 4B) = \text{rank}(M_{24} : 4C)$

*Proof.* We obtain that  $\xi_{M_{24}}(4A, 4A, 23A) = 1035$ ,  $\xi_{M_{24}}(4B, 4B, 23A) = 13501$  and  $\xi_{M_{24}}(4C, 4C, 23A) = 27508$ . Of the maximal subgroups  $M_{23}$  and  $L_2(23)$  of  $M_{24}$  which have elements of order 23,  $4a$  of  $M_{23}$  fuses into  $4B$  of  $M_{24}$  and  $4a$  of  $L_2(23)$  fuses into  $4C$  of  $M_{24}$  and none of them meets the  $4A$  class of  $M_{24}$ . Thus we have that  $\xi_{M_{23}}(4a, 4a, 23a) = 7866$  and  $\xi_{M_{L_2(23)}}(4a, 4a, 23a) = 23$ . By [7] and [14],  $23a$  of  $M_{23}$  and  $23a$  of  $L_2(23)$  are contained in one conjugate each of  $M_{23}$  and  $L_2(23)$ . Thus we obtain that  $\xi_{M_{24}}^*(4A, 4A, 23A) = 1035$ ,  $\xi_{M_{24}}^*(4B, 4B, 23A) = 13501 - 7866 = 5635$  and  $\xi_{M_{24}}^*(4C, 4C, 23A) = 27508 - 23 = 27485$ , so that  $M_{24}$  is  $(4A, 4A, 23A)$ ,  $(4B, 4B, 23A)$ ,  $(4C, 4C, 23A)$ -generated. Hence the result follows.  $\square$

Lemma 3.5 immediately following here below, will be used to prove some of the subsequent results.

**Lemma 3.5.** *Let  $G$  be a finite simple group such that  $G$  is  $(2X, mY, nZ)$ -generated. Then  $\text{rank}(G : mY) = 2$ .*

*Proof.* The result follows immediately from Corollary 2.2 above.  $\square$

**Proposition 3.6.**  $\text{rank}(M_{24} : 5A) = 2$

*Proof.* We obtained from [13] that  $M_{24}$  is  $(2B, 5A, 7A)$ -generated. Thus by Lemma 3.5 above, the result follows immediately and the proof is complete.  $\square$

**Proposition 3.7.**  $\text{rank}(M_{24} : 6A) = 2 = \text{rank}(M_{24} : 6B)$

*Proof.* We obtain that  $\xi_{M_{24}}(6A, 6A, 23A) = 427041$  and  $\xi_{M_{24}}(6B, 6B, 23A) = 406456$ . We obtain that  $6a$  of  $M_{23}$  fuses into  $6A$  of  $M_{24}$  whereas  $6a$  of  $L_2(23)$  fuses into  $6B$  of  $M_{24}$ . Thus we get that  $\xi_{M_{23}}(6a, 6a, 23a) = 72588$  and  $\xi_{M_{L_2(23)}}(6a, 6a, 23a) = 23$  so that  $\xi_{M_{24}}^*(6A, 6A, 23A) = 427041 - 72588 = 354453$  and  $\xi_{M_{24}}^*(6B, 6B, 23A) = 406456 - 23 = 406433$ . Thus we obtain that  $M_{24}$  is  $(6A, 6A, 23A)$ ,  $(6B, 6B, 23A)$ -generated. Hence the result follows thus completing the proof.  $\square$

**Proposition 3.8.**  $\text{rank}(M_{24} : 7A) = 2 = \text{rank}(M_{24} : 7B)$

*Proof.* In [13] it has been proved that  $M_{24}$  is  $(2, 7, 23)$ -generated and so the result follows immediately by Lemma 3.5 above and the proof is complete.  $\square$

**Proposition 3.9.**  $\text{rank}(M_{24} : 8A) = 2$

*Proof.* We obtain that  $\xi_{M_{24}}(8A, 8A, 23A) = 922024$  and of the maximal subgroups of  $M_{24}$  having elements of order 23, only  $M_{23}$  meets the  $8A$  class of  $M_{24}$ . So we obtain that  $\xi_{M_{23}}(8a, 8a, 23a) = 154376$  so that  $\xi_{M_{24}}^*(8A, 8A, 23A) = 922024 - 154376 = 767648$ . Thus we obtain that  $M_{24}$  is  $(8A, 8A, 23A)$ -generated and hence the result follows completing the proof.  $\square$

**Proposition 3.10.**  $\text{rank}(M_{24} : 10A) = 2$

*Proof.* We obtain that  $\xi_{M_{24}}(10A, 10A, 23A) = 607752$  and of the maximal subgroups of  $M_{24}$  having elements of order 23, none meets the  $10A$  class of  $M_{24}$ . Thus we obtain that  $\xi_{M_{24}}^*(10A, 10A, 23A) = 607752$  so that  $M_{24}$  is  $(10A, 10A, 23A)$ -generated and the result follows.  $\square$

**Proposition 3.11.**  $rank(M_{24} : 11A) = 2$

*Proof.* From [13] we have that  $M_{24}$  is  $(2, 11, 23)$ -generated. Therefore by Lemma 3.5 above, the result follows immediately and the proof is complete.  $\square$

**Proposition 3.12.**  $rank(M_{24} : 12A) = 2 = rank(M_{24} : 12B)$

*Proof.* We obtain that  $\xi_{M_{24}}(12A, 12A, 23A) = 1706232$  and  $\xi_{M_{24}}(12B, 12B, 23A) = 1625824$ . Only the maximal subgroup  $L_2(23)$  meets the  $12B$  class of  $M_{24}$  i.e. both  $12a$  and  $12b$  of  $L_2(23)$  fuse into the  $12B$  class of  $M_{24}$ . We thus obtain that  $\xi_{M_{L_2(23)}}(12a, 12a, 23a) = 23 = \xi_{M_{L_2(23)}}(12b, 12b, 23a)$ . So  $\xi_{M_{24}}^*(12A, 12A, 23A) = 1706232$  and  $\xi_{M_{24}}^*(12B, 12B, 23A) = 1625824 - 46 = 1625778$  so that  $M_{24}$  is  $(12A, 12A, 23A), (12B, 12B, 23A)$ -generated. Hence the result follows.  $\square$

**Proposition 3.13.**  $rank(M_{24} : 14A) = 2 = rank(M_{24} : 14B)$

*Proof.* We obtain that  $\xi_{M_{24}}(14A, 14A, 23A) = 1328940 = \xi_{M_{24}}(14B, 14B, 23A)$ . We have that  $14a$  and  $14b$  of  $M_{23}$  fuse into  $14A$  and  $14B$  of  $M_{24}$  respectively. Thus we obtain that  $\xi_{M_{23}}(14a, 14a, 23a) = 52992 = \xi_{M_{23}}(14b, 14b, 23a)$  so that  $\xi_{M_{24}}^*(14A, 14A, 23A) = 1328940 - 52992 = 1275948 = \xi_{M_{24}}^*(14B, 14B, 23A)$ . So  $M_{24}$  is  $(14A, 14A, 23A), (14B, 14B, 23A)$ -generated. Hence the result follows.  $\square$

**Proposition 3.14.**  $rank(M_{24} : 15A) = 2 = rank(M_{24} : 15B)$

*Proof.* We obtain that  $\xi_{M_{24}}(15A, 15A, 23A) = 1051008 = \xi_{M_{24}}(15B, 15B, 23A)$ . We have that  $15a$  and  $15b$  of  $M_{23}$  fuse into  $15A$  and  $15B$  of  $M_{24}$  respectively. Thus we obtain that  $\xi_{M_{23}}(15a, 15a, 23a) = 41998 = \xi_{M_{23}}(15b, 15b, 23a)$  so that  $\xi_{M_{24}}^*(15A, 15A, 23A) = 1051008 - 41998 = 1009010 = \xi_{M_{24}}^*(15B, 15B, 23A)$ . So  $M_{24}$  is  $(15A, 15A, 23A), (15B, 15B, 23A)$ -generated and hence the result follows.  $\square$

**Proposition 3.15.**  $rank(M_{24} : 21A) = 2 = rank(M_{24} : 21B)$

*Proof.* We obtain that  $\xi_{M_{24}}(21A, 21A, 23A) = 590640 = \xi_{M_{24}}(21B, 21B, 23A)$ . In this case there is no contribution from any of the maximal subgroups of  $M_{24}$  containing elements of order 23. This then renders  $\xi_{M_{24}}^*(21A, 21A, 23A) = 590640 = \xi_{M_{24}}^*(21B, 21B, 23A)$  so that  $M_{24}$  is  $(21A, 21A, 23A), (21B, 21B, 23A)$ -generated. Hence the result follows.  $\square$

**Proposition 3.16.**  $rank(M_{24} : 23A) = 2 = rank(M_{24} : 23B)$

*Proof.* We obtain that  $\xi_{M_{24}}(23A, 23A, 23A) = 441904$  and  $\xi_{M_{24}}(23B, 23B, 23A) = 455728$ . We have that  $23a$  and  $23b$  of both  $M_{23}$  and  $L_2(23)$  fuse into  $23A$  and  $23B$  of  $M_{24}$  respectively. Their contributions are thus  $\xi_{M_{23}}(23a, 23a, 23a) = 17646$ ,  $\xi_{M_{23}}(23b, 23b, 23a) = 18222$  and  $\xi_{M_{L_2(23)}}(23a, 23a, 23a) = 5$ ,  $\xi_{M_{L_2(23)}}(23b, 23b, 23a) = 29$ . Thus we obtain that  $\xi_{M_{24}}^*(23A, 23A, 23A) = 441904 - 17646 - 5 = 424253$  and  $\xi_{M_{24}}^*(23B, 23B, 23A) = 455728 - 18222 - 29 = 437477$ . These then render  $M_{24}$  to be  $(23A, 23A, 23A), (23B, 23B, 23A)$ -generated which thus give the desired results.  $\square$

We thus observe from all the proofs appearing here above that  $rank(M_{24} : nX) \in \{2, 3\}$  for any conjugacy class  $nX$  of nonidentity elements of  $M_{24}$ .

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