SOME TOPOLOGICAL INDICES AND GRAPH PROPERTIES

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Abstract. In this paper, by using the degree sequences of graphs, we present sufficient conditions for a graph to be Hamiltonian, traceable, Hamilton-connected or $k$-connected in light of numerous topological indices such as the eccentric connectivity index, the eccentric distance sum, the connective eccentricity index.

1. Introduction

Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V| = n$ and $|E| = m$. Let $d(v)$ be the degree of a vertex $v$ in $G$. Let $d(u, v)$ be the distance between two vertices $u$ and $v$ in $G$, that is, the length of the shortest path connecting $u$ and $v$ in $G$. The eccentricity $\varepsilon(v)$ of a vertex $v$ is the maximum distance from $v$ to any other vertex. Let $K_n, S_n, P_n$ be a complete graph, a star and a path on $n$ vertices, respectively.

A cycle $C$ in a graph $G$ is called a Hamiltonian cycle of $G$ if $C$ contains all the vertices of $G$. A graph $G$ is called Hamiltonian if $G$ has a Hamiltonian cycle. A path $P$ in a graph $G$ is called a Hamiltonian path of $G$ if $P$ contains all the vertices of $G$. A graph $G$ is called traceable if $G$ has a Hamiltonian path. A graph $G$ is called Hamilton-connected if for each pair of vertices in $G$ there is a Hamiltonian path between them. A graph $G$ is said to be $k$-connected (or $k$-vertex connected) if there does not exist a set of $k - 1$ vertices whose removal disconnects the graph. If $G$ and $H$ are two vertex-disjoint graphs, we use $G \vee H$ to denote the join of $G$ and $H$.

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The topological indices are widely used in organic chemistry and have been found to be useful in chemical documentation, isomer discrimination, structure-property relationships, structure-activity (SAR) relationships and pharmaceutical drug design [14, 23]. In past decades, plenty of mathematical properties of numerous topological indices are reported such as the matching energy [5, 6], Randić index [24] and the Balaban index [7].

For a connected graph $G$, its Wiener index, denoted by $W(G)$, is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v) = \frac{1}{2} \sum_{v \in V(G)} D(v).$$

Here $D(v) = \sum_{u \in V(G)} d_{G}(u, v)$. It can be easily verified that $D(v) \geq d(v) + 2(n - 1 - d(v))$. The Wiener index and its modifications are well studied in the past years, see [9, 17, 21, 19, 20].

The eccentric connectivity index (ECI) [22] of a connected graph $G$, denoted by $\xi^{c}(G)$, is defined as

$$\xi^{c}(G) = \sum_{v \in V(G)} \varepsilon(v) d(v).$$

The eccentric distance sum (EDS) [11] of a connected graph $G$ is defined as

$$\xi^{d}(G) = \sum_{v \in V(G)} \varepsilon(v) \cdot D(v).$$

The connective eccentricity index (CEI) [10] of a connected graph $G$ is defined as

$$\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)}.$$

The above three topological indices involving eccentricities are widely studied from mathematical view, see [13, 18, 26, 27, 28].

In [25], Yang presented a sufficient condition for a graph to be traceable by using Wiener index. In [12], Hua and Wang presented a sufficient condition for a graph to be traceable by using Harary index. Li [15, 16] presented sufficient conditions in terms of the Harary index and Wiener index for a graph to be Hamiltonian or Hamilton-connected using some proof ideas in [25].

In this paper, as a continuance of the above results, we further study the conditions for a graph to be Hamiltonian, traceable, Hamilton-connected or $k$-connected in light of numerous topological indices such as the ECI, EDS and CEI.

2. Preliminaries

We first present some lemmas that will be used later.

Lemma 2.1. [8] Let $G$ be a graph of order $n \geq 3$ with degree sequence $d_{1} \leq d_{2} \leq \cdots \leq d_{n}$. If $d_{k} \leq k < \frac{n}{2} \Rightarrow d_{n-k} \geq n - k$, then $G$ is Hamiltonian.

Lemma 2.2. [1] Let $G$ be a nontrivial graph of order $n \geq 4$ with degree sequence $d_{1} \leq d_{2} \leq \cdots \leq d_{n}$. If $d_{k} + 1 \leq k < \frac{n+1}{2} \Rightarrow d_{n-k+1} \leq n - k - 1$, then $G$ is traceable.
Lemma 2.3. [3] Let $G$ be a graph of order $n \geq 4$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $d_i \leq i + k - 2 \Rightarrow d_{n-k+1} \geq n - i$, for $1 \leq i \leq \frac{1}{2}(n - k + 1)$, then $G$ is $k$-connected.

Lemma 2.4. [2] Let $G$ be a graph of order $n \geq 4$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $2 \leq k \leq \frac{n}{2}, d_{k-1} \leq k \Rightarrow d_{n-k} \geq n - k + 1$, then $G$ is Hamilton-connected.

Lemma 2.5. [2, Page 210, Corollary 5] Let $G = (X,Y; E)$ be a bipartite graph such that $X = \{x_1,x_2,\ldots,x_n\}$, $Y = \{y_1,y_2,\ldots,y_n\}$, $n \geq 2$, and $d(x_1) \leq d(x_2) \leq \cdots \leq d(x_n)$, $d(y_1) \leq d(y_2) \leq \cdots \leq d(y_n)$. If $d(x_k) \leq k < n \Rightarrow d(y_{n-k}) \geq n - k + 1$, then $G$ is Hamiltonian.

Lemma 2.6. [4] Let $G$ be a 2-connected graph of order $n \geq 12$. If $m \geq \binom{n-2}{2} + 4$, then $G$ is Hamiltonian or $G = K_2 \vee ((2K_1) \cup K_{n-4})$.

Lemma 2.7. [4] Let $G$ be a 3-connected graph of order $n \geq 18$. If $m \geq \binom{n-3}{2} + 9$, then $G$ is Hamiltonian or $G = K_3 \vee ((3K_1) \cup K_{n-6})$.

Lemma 2.8. [4] Let $G$ be a $k$-connected graph of order $n$. If $m \geq \binom{n}{2} - (k+1)(n-k-1)/2 + 1$, then $G$ is Hamiltonian.

3. Main Results

Theorem 3.1. Let $G$ be a connected graph of order $n \geq 6$.

1. If $\xi^e(G) \geq n^3 - 3n^2 + 4n - \frac{4m^2}{n} > 0$, then $G$ is Hamiltonian.

2. If $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + n - 4)^2$, then $G$ is Hamiltonian.

3. If $\xi^c(G) \geq (n-1)\frac{n^2 - 3n + 5}{n}$, then $G$ is Hamiltonian.

Proof. Suppose that $G$ is not Hamiltonian, then from Lemma 2.1, there exists an integer $1 \leq k \leq \frac{n-1}{2}$ such that $d_k \leq k$ and $d_{n-k} \leq n - k - 1$.

1. We consider $\xi^e(G)$. Since $c(v) \leq v - d(v)$, from the definition, we have

\[
\xi^e(G) = \sum_{v \in V(G)} c(v)d(v) \leq \sum_{v \in V(G)} (n - d(v))d(v)
\]

\[
= n\left(\sum_{v \in V(G)} d(v)\right) - \sum_{v \in V(G)} d^2(v)
\]

\[
\leq n\left(\sum_{v \in V(G)} d(v)\right) - \frac{1}{n}\left(\sum_{v \in V(G)} d(v)\right)^2
\]

\[
= n\left(\sum_{v \in V(G)} d(v)\right) - \frac{4m^2}{n}
\]

\[
\leq n\left[k^2 + (n - 2k)(n - k - 1) + k(n - 1)\right] - \frac{4m^2}{n}
\]

\[
= n^2(n - 1) + n\left[3k^2 - (2n - 1)k\right] - \frac{4m^2}{n}.
\]
Suppose \( f(x) = 3x^2 - (2n-1)x \) with \( 1 \leq x \leq \frac{1}{2}(n-1) \). It is easy to see that \( f_{\max}(x) = \max\{f(1), f(\frac{1}{2}(n-1))\} \). As \( f(1) = 4-2n, f(\frac{1}{2}(n-1)) = \frac{1}{2}(1-n^2), f(\frac{n-1}{2}) - f(1) = -\frac{1}{4}(n-5)(n-3) < 0 \), so we have \( f_{\max}(x) = f(1) \). Thus, \( \xi^c(G) \leq n^2(n-1) + n(4-2n) - \frac{4m^2}{n} = n^3 - 3n^2 + 4n - \frac{4m^2}{n} \), so we get the result.

If \( \xi^c(G) = n^3 - 3n^2 + 4n - \frac{4m^2}{n} \), then all the inequalities in the proof should be equalities, so \( k = 1 \), and hence \( d_1 = 1, d_2 = d_3 = \cdots = d_{n-1} = n-2, d_n = n-1 \). Thus \( G = K_1 \lor (K_1 \lor K_{n-2}) \), which is not Hamiltonian as stated in [1]. But this graph does not satisfy \( \sum_{v \in V(G)} d^2(v) = \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2 \), thus the equality can not hold.

(2) We consider \( \xi^d(G) \). Since \( \varepsilon(v) \geq \frac{D(v)}{n-1}, D(v) \geq d(v) + 2(n-1 - d(v)) \), from the definition, we have

\[
\xi^d(G) = \sum_{v \in V(G)} \varepsilon(v) \cdot D(v) \geq \sum_{v \in V(G)} \frac{D(v)}{n-1} \cdot D(v)
\]
\[
= \frac{1}{n-1} \sum_{v \in V(G)} (D(v))^2
\]
\[
\geq \frac{1}{n-1} \sum_{v \in V(G)} \left[ 4(n-1)^2 - 4(n-1)d(v) + (d(v))^2 \right]
\]
\[
= 4n(n-1) - 4 \sum_{v \in V(G)} d(v) + \frac{1}{n-1} \sum_{v \in V(G)} (d(v))^2
\]
\[
\geq 4n(n-1) - 4 \sum_{v \in V(G)} d(v) + \frac{1}{n-1} \cdot \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2
\]
\[
= \frac{1}{n(n-1)} \left[ \left( \sum_{v \in V(G)} d(v) \right)^2 - 4n(n-1) \sum_{v \in V(G)} d(v) + 4n^2(n-1)^2 \right]
\]
\[
= \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2 .
\]

As \( \sum_{v \in V(G)} d(v) < 2n(n-1) \), we have

\[
\xi^d(G) \geq \frac{1}{n(n-1)} \left[ k^2 + (n-2k)(n-k-1) + k(n-1) - 2n(n-1) \right]^2
\]
\[
= \frac{1}{n(n-1)} \left[ 2n(n-1) - [k^2 + (n-2k)(n-k-1) + k(n-1)] \right]^2
\]
\[
= \frac{1}{n(n-1)} \left[ -3k^2 + (2n-1)k + n^2 - n \right]^2 .
\]

Suppose \( f(x) = -3x^2 + (2n-1)x + n^2 - n \) with \( 1 \leq x \leq \frac{1}{2}(n-1) \). As \( f(1) = n^2 + n - 4, f(\frac{n-1}{2}) = \frac{1}{4}(n-1)(5n+1), f(\frac{n-1}{2}) - f(1) = \frac{1}{4}(n-3)(n-5) > 0 \), so we have \( f_{\min}(x) = \min\{f(1), f(\frac{1}{2}(n-1))\} = f(1) \). Thus \( \xi^d(G) \geq \frac{1}{n(n-1)} (n^2 + n - 4)^2 \), and we get the result.

If \( \xi^d(G) = \frac{1}{n(n-1)} (n^2 + n - 4)^2 \), then \( k = 1 \), the remaining is as in the previous proof.
(3) We consider $\xi^{ce}(G)$. From the definition, we have

$$
\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)}
$$

$$
\leq \sum_{v \in V(G)} \frac{n-1}{D(v)} \cdot d(v)
$$

$$
\leq (n-1) \sum_{v \in V(G)} \frac{d(v)}{2(n-1) - d(v)}.
$$

Suppose $f(x) = \frac{x}{2(n-1)-x}$, then we have $f'(x) = \frac{2(n-1)}{[2(n-1)-x]^2} > 0$, and thus $f(x)$ is strictly increasing, therefore

$$
\xi^{ce}(G) \leq (n-1) \left[ \frac{k^2}{2(n-1) - k} + \frac{(n-2k)(n-k-1)}{2(n-1) - (n-k-1)} + \frac{k(n-1)}{2(n-1) - (n-1)} \right]
$$

$$
= (n-1) \left[ \frac{k^2}{2n-k-2} + \frac{(n-2k)(n-k-1)}{n+k-1} \right].
$$

Since $1 \leq k \leq \frac{n-1}{2}$, then $2n - k - 2 - (n + k - 1) = n - 2k - 1 \geq 0$, so $\frac{k^2}{2n-k-2} \leq \frac{k^2}{n+k-1}$. Further, $\frac{(n-2k)(n-k-1)}{n+k-1} = n - 2k - \frac{2k(n-2k)}{n+k-1}$. Therefore,

$$
\xi^{ce}(G) \leq (n-1) \left[ \frac{k^2}{n+k-1} - \frac{2k(n-2k)}{n+k-1} - k + n \right]
$$

$$
= (n-1) \left[ \frac{k^2}{n+k-1} - k + n \right]
$$

$$
= (n-1) \left\{ \frac{k[4k-(3n-1)]}{n+k-1} + n \right\}.
$$

Suppose $f(x) = \frac{x[4x-(3n-1)]}{n+x-1}$ with $1 \leq x \leq \frac{1}{2}(n-1)$. As $f(1) = \frac{5-3n}{n}$, $f(\frac{n-1}{2}) = -\frac{1}{3n}(n-3)(n-5) < 0$, so we have $f_{\max}(x) = \max\{f(1), f(\frac{1}{2}(n-1))\} = f(1)$. Thus $\xi^{ce}(G) \leq (n-1)\frac{n^2-3n+5}{n}$, and we get the result.

If $\xi^{ce}(G) = (n-1)\frac{n^2-3n+5}{n}$, then all the inequalities in the proof should be equalities, so $k = 1$, and hence $d_1 = 1$, $d_2 = d_3 = \cdots = d_{n-1} = n - 2$, $d_n = n - 1$. Thus $G = K_1 \cup (K_1 \cup K_{n-2})$, which is not Hamiltonian as stated in [1].

On the other hand, $\varepsilon(v) \geq \frac{D(v)}{n-1}$, with equality if and only if $d(v,u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \neq u$. Thus, $G = K_1 \cup (K_1 \cup K_{n-2})$ can not satisfy it, the equality can not hold.

\[ \square \]

**Theorem 3.2.** Let $G$ be a connected graph of order $n \geq 11$.

1. If $\xi^c(G) \geq n^3 - 5n^2 + 10n - \frac{4n^2}{n} > 0$, then $G$ is traceable.
2. If $\xi^c(G) \leq \frac{1}{n(n-1)}(n^2 + 3n - 10)^2$, then $G$ is traceable.
3. If $\xi^{ce}(G) \geq (n-1)\frac{n^2-5n+12}{n+1}$, then $G$ is traceable.

**Proof.** Suppose that $G$ is not traceable, then by Lemma 2.2, there is an integer $k \leq \frac{n}{2}$ such that $d_k \leq k - 1$ and $d_{n-k+1} \leq n - k - 1$. Since $G$ is connected and $d_k \leq k - 1$, we have $k \geq 2$. 

(1) We consider $\xi^c(G)$. As in Theorem 3.1, we have
\[
\xi^c(G) \leq n \left( \sum_{v \in V(G)} d(v) \right) - \frac{4m^2}{n} \\
\leq n [k(k-1) + (n-2k+1)(n-k-1) + (k-1)(n-1)] - \frac{4m^2}{n} \\
= n^2(n-1) - \frac{4m^2}{n} + n [3k^2 - (2n+1)k].
\]

Suppose $f(x) = 3x^2 - (2n+1)x$ with $2 \leq x \leq \frac{n}{2}$. As $f(2) = 10 - 4n$, $f(\frac{n}{2}) = -\frac{1}{4}n(n+2)$, $f(\frac{n}{2}) - f(2) = -\frac{1}{4}(n-10)(n-4) < 0$, so we have $f_{\max}(x) = \max\{f(2), f(\frac{n}{2})\} = f(2)$. Thus
\[
\xi^c(G) \leq n^2(n-1) - \frac{4m^2}{n} + n [3k^2 - (2n+1)k],
\]
where $n = 3$. As in Theorem 3.1, we have
\[
\xi^c(G) = n^3 - 5n^2 + 10n - \frac{4m^2}{n},
\]
so we get the result.

If $\xi^c(G) = n^3 - 5n^2 + 10n - \frac{4m^2}{n}$, then $k = 2$, and hence $d_1 = d_2 = 1$, $d_3 = \cdots = d_{n-1} = n-3$, $d_n = n-1$. Thus $G = K_1 \cup (K_{n-3} \cup 2K_1)$, which is not traceable. But this graph does not satisfy
\[
\sum_{v \in V(G)} d^2(v) = \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2,
\]
thus the equality cannot hold.

(2) We consider $\xi^d(G)$, as in Theorem 3.1, we have
\[
\xi^d(G) \geq \frac{1}{n(n-1)} \left( \sum_{v \in V(G)} d(v) - 2n(n-1) \right)^2.
\]

Since $2n(n-1) - \sum_{v \in V(G)} d(v) > 0$, we have
\[
\xi^d(G) \geq \frac{1}{n(n-1)} \left[ k(k-1) + (n-2k+1)(n-k-1) + (k-1)(n-1) - 2n(n-1) \right]^2 \\
= \frac{1}{n(n-1)} \left[ 2n(n-1) - [k(k-1) + (n-2k+1)(n-k-1) + (k-1)(n-1)] \right]^2 \\
= \frac{1}{n(n-1)} \left[ -3k^2 + (2n+1)k + n(n-1) \right]^2.
\]

Suppose $f(x) = -3x^2 + (2n+1)x + n^2 - n$ with $2 \leq x \leq \frac{n}{2}$. As $f(2) = n^2 + 3n - 10$, $f(\frac{n}{2}) = \frac{1}{4}n(5n-2)$, $f(\frac{n}{2}) - f(2) = \frac{1}{4}(n-4)(n-10) \geq 0$, so we have $f_{\min}(x) = \min\{f(2), f(\frac{n}{2})\} = f(2)$. Thus
\[
\xi^d(G) \geq \frac{1}{n(n-1)} (n^2 + 3n - 10)^2,
\]
so we get the result.

If $\xi^d(G) = \frac{1}{n(n-1)} (n^2 + 3n - 10)^2$, then $k = 2$, the remaining is as in the previous proof.

(3) We consider $\xi^{ce}(G)$. As in Theorem 3.1,
\[
\xi^{ce}(G) \leq (n-1) \sum_{v \in V(G)} \frac{d(v)}{2(n-1) - d(v)}.
\]

Suppose $f(x) = \frac{x}{2(n-1)-x}$, then $f'(x) = \frac{2(n-1)}{(2(n-1)-x)^2} > 0$, so
\[
\xi^{ce}(G) \leq (n-1) \left[ \frac{k(k-1)}{2(n-1) - (k-1)} + \frac{(n-2k+1)(n-k-1)}{2(n-1) - (n-k-1)} + \frac{(k-1)(n-1)}{2(n-1) - (n-1)} \right] \\
= (n-1) \left[ \frac{k(k-1)}{2n-k-1} + \frac{(n-2k+1)(n-k-1)}{n+k-1} + k-1 \right].
\]
Since \(2 \leq k \leq \frac{n}{2}\), then \(2n - k - 1 - (n + k - 1) = n - 2k \geq 0\). As \(\frac{(n-2k+1)(n-k-1)}{n+k-1} = \frac{(n-2k+1)(n-k-2k)}{n+k-1}\), therefore,

\[
\xi^c(G) \leq (n-1) \left[ \frac{k(k-1)}{n+k-1} - \frac{2k(n-2k+1)}{n+k-1} - k + n \right]
= (n-1) \left[ \frac{k(k-1) - 2k(n-2k+1)}{n+k-1} - k \right]
= (n-1) \left\{ \frac{k[4k - (3n + 2)]}{n+k-1} + n \right\}.
\]

Suppose \(f(x) = \frac{x[4x - (3n + 2)]}{n + x - 1}\) with \(2 \leq x \leq \frac{n}{2}\). It is easy to see that \(f_{\max}(x) = \max\{f(2), f(\frac{n}{2})\}\). As \(f(2) = \frac{6(2-n)}{n+1}\), \(f(\frac{n}{2}) = -\frac{n(n+2)}{3n-2}\), \(f(2) = -\frac{(n-4)(n^2-11n+6)}{(3n-2)(n+1)} < 0\), so we have \(f_{\max}(x) = f(2)\). Thus \(\xi^c(G) \leq (n-1) \frac{2^2 - 5n + 12}{n+1}\), so we get the result.

If \(\xi^c(G) = (n-1) \frac{2^2 - 5n + 12}{n+1}\), then \(k = 2\), and hence \(d_1 = d_2 = 1\), \(d_3 = \cdots = d_{n-1} = n-3\), \(d_n = n-1\). Thus \(G = K_1 \cup (K_{n-3} \cup 2K_1)\), which is not traceable.

On the other hand, \(\varepsilon(v) \geq \frac{2\nu(v)}{n-1}\), with equality if and only if \(d(v, u)\) (for fixed \(v \in V(G)\)) is a constant for all \(u \in V(G)\) with \(v \neq u\). But \(G = K_1 \cup (K_{n-3} \cup 2K_1)\) can not satisfy it, and the equality case can not occur.

\[\square\]

**Theorem 3.3.** Let \(G\) be a connected graph of order \(n \geq 2\).

1. If \(\xi^c(G) \geq n^3 - 3n^2 + 2kn - \frac{4m^2}{n} > 0\), then \(G\) is \(k\)-connected.
2. If \(\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + n - 2k)^2\), then \(G\) is \(k\)-connected.
3. If \(\xi^c(G) \geq (n-1)\left(\frac{3k-3n-1}{n} + n\right)\), then \(G\) is \(k\)-connected.

**Proof.** Suppose that \(G\) is not \(k\)-connected, then from Lemma 2.3, there exists an integer \(1 \leq i \leq \frac{n-k+1}{2}\) such that \(d_i \leq i + k - 2\) and \(d_{n-k+1} \leq n - i - 1\). Obviously, \(1 \leq k \leq n - 1\).

(1) We consider \(\xi^c(G)\), as in Theorem 3.1, we have

\[
\xi^c(G) \leq n \left( \sum_{v \in V(G)} d(v) \right) - \frac{4m^2}{n}
\leq n \left[ i(i + k - 2) + (n - k - i + 1)(n - i - 1) + (k - 1)(n - 1) \right] - \frac{4m^2}{n}
= n^2(n - 1) - \frac{4m^2}{n} + 2n \left[ i^2 - (n - k + 1)i \right] .
\]

Suppose \(f(x) = x^2 - (n - k + 1)x\) with \(1 \leq x \leq \frac{n-k+1}{2}\), then \(f(x) \leq f(1) = k - n\). Thus \(\xi^c(G) \leq n^2(n - 1) - \frac{4m^2}{n} + 2n(k - n) = n^3 - 3n^2 + 2kn - \frac{4m^2}{n}\), so we get the result.

If \(\xi^c(G) = n^3 - 3n^2 + 2kn - \frac{4m^2}{n}\), then all the inequalities in the proof should be equalities, so \(i = 1\), \(d_1 = k - 1\), \(d_2 = \cdots = d_{n-k+1} = n - 2\), \(d_{n-k+2} = \cdots = d_n = n - 1\). Thus \(G = (K_1 \cup K_{n-k}) \cup K_{k-1}\), which is not \(k\)-connected. But it can not satisfy \(\sum_{v \in V(G)} d^2(v) = \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2\), thus the equality can not hold.
(2) We consider $\xi^d(G)$. As in Theorem 3.1,

$$
\xi^d(G) \geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2.
$$

Since $2n(n-1) - \sum_{v \in V(G)} d(v) > 0$, then

$$
\xi^d(G) \geq \frac{1}{n(n-1)} \left[ i(i+k-2) + (n-k-i+1)(n-i-1) + (k-1)(n-1) - 2n(n-1) \right]^2
\geq \frac{1}{n(n-1)} \left[ 2n(n-1) - [i(i+k-2) + (n-k-i+1)(n-i-1) + (k-1)(n-1)] \right]^2
= \frac{1}{n(n-1)} \left[ -2i^2 - 2i(1+k-n) + n(n-1) \right]^2.
$$

Suppose $f(x) = -2x^2 - 2x(-1+k-n) + n(n-1)$ with $1 \leq x \leq \frac{n-k+1}{2}$, $f(1) \leq f(x) \leq f\left(\frac{n-k+1}{2}\right)$, $f(1) = n(n+1) - 2k$. Thus $\xi^d(G) \geq \frac{1}{n(n-1)}(n(n+1) - 2k)^2$, so we get the result.

If $\xi^d(G) = \frac{1}{n(n-1)}(n^2 + n - 2k)^2$, then all the inequalities in the proof should be equalities, so $i = 1$, the remaining is as in the previous proof.

(3) We consider $\xi^{ce}(G)$, as in Theorem 3.1, we have

$$
\xi^{ce}(G) \leq (n-1) \sum_{v \in V(G)} \frac{d(v)}{2(n-1) - d(v)}.
$$

Suppose $f(x) = \frac{x}{2(n-1)-x}$, $f'(x) = \frac{2(n-1)}{(2(n-1)-x)^2} > 0$, so

$$
\xi^{ce}(G) \leq (n-1) \left[ \frac{i(i+k-2)}{2(n-1) - (i+k-2)} + \frac{(n-k-i+1)(n-i-1)}{2(n-1) - (n-i-1)} + \frac{(k-1)(n-1)}{2(n-1) - (n-1)} \right]
= (n-1) \left[ \frac{i(i+k-2)}{2n-k+i} + \frac{(n-k-i+1)(n-i-1)}{n+i-1} + k-1 \right].
$$

Since $1 \leq i \leq \frac{n-k+1}{2}$, then $2n-k-i-(n+i-1) = n-k-2i+1 \geq 0$. Further, $\frac{(n-k-i+1)(n-i-1)}{n+i-1} = n-k+i-1 - \frac{2(n-k-i+1)}{n+i-1}$.

Therefore,

$$
\xi^{ce}(G) \leq (n-1) \left[ \frac{i(i+k-2)}{n+i-1} - \frac{2i(n-k-i+1)}{n+i-1} - i + n \right]
= (n-1) \left[ \frac{i(i+k-2) - 2i(n-k-i+1)}{n+i-1} - i + n \right]
= (n-1) \left[ \frac{i(2i+3k-3n-3)}{n+i-1} + n \right].
$$

Suppose $f(x) = \frac{x(2x+3k-3n-3)}{n+x-1}$ with $1 \leq x \leq \frac{n-k+1}{2}$, we can easily compute that $f_{\text{max}}(x) = f(1) = \frac{3k-3n-1}{n}$. Thus $\xi^{ce}(G) \leq (n-1)\left(\frac{3k-3n-1}{n} + n\right)$, so we get the result.

If $\xi^{ce}(G) = (n-1)\left(\frac{3k-3n-1}{n} + n\right)$, then $i = 1$, $d_1 = k-1$, $d_2 = \cdots = d_{n-k+1} = n-2$, $d_{n-k+2} = \cdots = d_n = n-1$. Thus $G = (K_1 \cup K_{n-k}) \cup K_{k-1}$, which is not $k$-connected.
On the other hand, $\varepsilon(v) \geq \frac{D(v)}{n-1}$, with equality if and only if $d(v, u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \neq u$. But $G = (K_1 \cup K_{n-k}) \cup K_{k-1}$ can not satisfy it, the equality can not hold.

\[\square\]

Theorem 3.4. Let $G$ be a connected graph of order $n \geq 7$.

1. If $\xi^c(G) \geq n^3 - 3n^2 + 6n - \frac{4m^2}{n} > 0$, then $G$ is Hamilton-connected.
2. If $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + n - 6)^2$, then $G$ is Hamilton-connected.
3. If $\xi^{m^e}(G) \geq (n-1)(\frac{3}{n} + n - 3)$, then $G$ is Hamilton-connected.

Proof. Suppose that $G$ is not Hamilton-connected, then from Lemma 2.4, there exists an integer $2 \leq k \leq \frac{n}{2}$ such that $d_{k-1} \leq k$ and $d_{n-k} \leq n - k$.

1. We consider $\xi^c(G)$, as in Theorem 3.1, we have

$$\xi^c(G) \leq n \left( \sum_{v \in V(G)} d(v) \right) - \frac{4m^2}{n}$$

$$\leq n [(k-1)k + (n-2k+1)(n-k) + k(n-1)] - \frac{4m^2}{n}$$

$$= n^2(n+1) - \frac{4m^2}{n} + n \left[3k^2 - (2n+3)k \right] .$$

Suppose $f(x) = 3x^2 - (2n+3)x$ with $2 \leq x \leq \frac{n}{2}$. As $f(2) = 6 - 4n$, $f \left( \frac{n}{2} \right) = \frac{1}{4}n(n+6)$, $f \left( \frac{n}{2} \right) - f(2) = \frac{1}{4}(n-6)(n-4) < 0$, so we have $f_{\text{max}}(x) = \text{max} \{f(2), f \left( \frac{n}{2} \right) \} = f(2)$. Thus, $\xi^c(G) \leq n^2(n+1) - \frac{4m^2}{n} + n[6 - 4n] = n^3 - 3n^2 + 6n - \frac{4m^2}{n}$, so we get the result.

If $\xi^c(G) = n^3 - 3n^2 + 6n - \frac{4m^2}{n}$, then $k = 2$, $d_1 = 2$, $d_2 = d_3 = \cdots = d_{n-2} = n - 2$, $d_{n-1} = d_n = n - 1$. Thus $G = K_2 \cup (K_1 \cup K_{n-3})$, which is not Hamilton-connected. But it can not satisfy $\sum_{v \in V(G)} d^2(v) = \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2$, thus the equality can not hold.

2. We consider $\xi^d(G)$, as in Theorem 3.1, we have

$$\xi^d(G) \geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2 .$$

Since $2n(n-1) - \sum_{v \in V(G)} d(v) > 0$, then

$$\xi^d(G) \geq \frac{1}{n(n-1)} [k(k-1) + (n-2k+1)(n-k) + k(n-1) - 2n(n-1)]^2$$

$$= \frac{1}{n(n-1)} \left[ (n-1) - [k(k-1) + (n-2k+1)(n-k) + k(n-1)] \right]^2$$

$$= \frac{1}{n(n-1)} \left[ -3k^2 + (2n+3)k + n(n-3) \right]^2 .$$

Suppose $f(x) = -3x^2 + (2n+3)x + n(n-3)$ with $2 \leq x \leq \frac{n}{2}$. As $f(2) = n^2 + n - 6$, $f \left( \frac{n}{2} \right) - f(2) = \frac{1}{4}(n-4)(n-6) > 0$, so we have $f_{\text{min}}(x) = \text{min} \{f(2), f \left( \frac{n}{2} \right) \} = f(2)$. Thus $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2 + n - 6)^2$, so we get the result.

If $\xi^d(G) = \frac{1}{n(n-1)}(n^2 + n - 6)^2$, then $k = 2$, the remaining is as in the previous proof.
(3) We consider $\xi^{ce}(G)$, as in Theorem 3.1, we have

$$\xi^{ce}(G) \leq (n - 1) \sum_{v \in V(G)} \frac{d(v)}{2(n - 1) - d(v)}.$$ 

Suppose $f(x) = \frac{x}{2(n-1)-x}$, $f'(x) = \frac{2(n-1)x}{(2(n-1)-x)^2} > 0$, so

$$\xi^{ce}(G) \leq (n - 1) \left[ \frac{k(k-1)}{2(n-1)} + \frac{(n-2k+1)(n-k)}{2(n-1) - (n-k)} + \frac{k(n-1)}{2(n-1) - (n-1)} \right].$$

= (n - 1) \left[ \frac{k(k-1)}{2n-k-2} + \frac{(n-2k+1)(n-k)}{n+k-2} + k \right].$$

Since $2 \leq k \leq \frac{n}{2}$, then $2n - k - 2 - (n + k - 2) = n - 2k \geq 0$. Further, $\frac{(n-2k+1)(n-k)}{n+k-2} = n - 2k - 1 - \frac{(2k-2)(n-2k+1)}{n+k-2}$. Therefore,

$$\xi^{ce}(G) \leq (n - 1) \left\{ \frac{4k^2 - (3n+5)k + 2n + 2}{n+k-2} + n + 1 \right\}.$$ 

Suppose $f(x) = \frac{4x^2-(3n+5)x+2n+2}{n+x-2}$ with $2 \leq x \leq \frac{n}{2}$. As $f(\frac{n}{2}) = -\frac{n^2+4n-4}{n(n-4)}$, $f(2) = \frac{8}{n} - 4$, $f(\frac{n}{2}) - f(2) = -(\frac{n-4}{n^2-7n+8}) < 0$. So we have $f_{\text{max}}(x) = \max\{f(2), f(\frac{n}{2})\} = f(2)$. Thus $\xi^{ce}(G) \leq (n - 1)\{f(2) + n + 1\} = (n - 1)(\frac{8}{n} + n - 3)$, so we get the result.

If $\xi^{ce}(G) = (n - 1)(\frac{8}{n} + n - 3)$, then $k = 2$, $d_1 = 2$, $d_2 = d_3 = \cdots = d_{n-2} = n - 2$, $d_{n-1} = d_n = n - 1$. Thus $G = K_2 \cup (K_1 \cup K_{n-3})$, which is not Hamilton-connected.

On the other hand, $\varepsilon(v) \geq \frac{D(v)}{n-1}$, with equality if and only if $d(v, u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \neq u$. Thus, $G = K_2 \vee (K_1 \cup K_{n-3})$ can not satisfy it, the equality can not hold. 

\[\square\]

**Theorem 3.5.** Let $G = (X, Y; E)$ be a connected bipartite graph with $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and $n \geq 2$. Then we have:

1. If $\xi^{d}(G) \leq \frac{1}{n(n-1)}(n^2 + 2n - 4)^2 + \frac{n(n-2)^2}{n-1}$, then $G$ is Hamiltonian.
2. If $\xi^{ce}(G) \geq (n - 1) \left[ \frac{1}{6}(n^2 - 2n + 2) + \frac{n^2}{3n-2} \right]$, then $G$ is Hamiltonian.

**Proof.** Suppose that $G$ is not Hamiltonian, then from Lemma 2.5, there exists an integer $k < n$ such that $d(x_k) \leq k$ and $d(y_{n-k}) \leq n - k$. Let $N(x_1) := \{z_1, z_2, \ldots z_k\}$ be the neighbors of $x_1$, where $s = d(x_1)$. Then $d(x_i, z_i) = 1$ for each $z_i \in N(x_1)$, $d(x_1, x_i) \geq 2$ for each $x_i$ with $2 \leq i \leq n$, and $d(x_1, y_i) \geq 3$ for each $y_i \in Y - N(x_1)$.

1. We consider $\xi^{d}(G)$. First, we have

$$D(x_1) \geq d(x_1) + 2(n - 1) + 3(n - d(x_1)) = 5n - 2 - 2d(x_1).$$
Similarly, for each $i$ with $2 \leq i \leq n$ and each $j$ with $1 \leq j \leq n$,

$$D(x_i) \geq d(x_i) + 2(n-1) + 3(n - d(x_i)) = 5n - 2 - 2d(x_i).$$

$$D(y_j) \geq d(y_j) + 2(n-1) + 3(n - d(y_j)) = 5n - 2 - 2d(y_j).$$

Therefore, we have

$$\xi^d(G) = \sum_{v \in V(G)} \varepsilon(v) \cdot D(v) \geq \sum_{v \in V(G)} \frac{D(v)}{n-1} \cdot D(v) = \frac{1}{n-1} \sum_{v \in V(G)} (D(v))^2$$

$$= \frac{1}{n-1} \left[ \sum_{x_i \in X} (D(v_i)^2) + \sum_{y_j \in Y} (D(v_j)^2) \right]$$

$$\geq \frac{1}{n-1} \left\{ \sum_{x_i \in X} \left[ (5n - 2)^2 - 4(5n - 2)d(x_i) + 4(d(x_i))^2 \right] + \sum_{y_j \in Y} \left[ (5n - 2)^2 - 4(5n - 2)d(y_j) + 4(d(y_j))^2 \right] \right\}$$

$$= \frac{1}{n(n-1)} \left[ 2n^2(5n - 2)^2 - 4n(5n - 2) \sum_{v \in V(G)} d(v) + 4 \sum_{v \in V(G)} (d(v))^2 \right]$$

$$\geq \frac{1}{n(n-1)} \left[ 2n^2(5n - 2)^2 - 4n(5n - 2) \sum_{v \in V(G)} d(v) + 4 \left( \sum_{v \in V(G)} d(v) \right)^2 \right]$$

$$= \frac{1}{n(n-1)} \left[ n(5n - 2) - 2 \sum_{v \in V(G)} d(v) \right]^2 + \frac{n(5n - 2)^2}{n-1}.$$

Since

$$2 \sum_{v \in V(G)} d(v) \leq 2 \left[ k^2 + (n-k)n + (n-k)^2 + kn \right]$$

$$\leq 2 \left[ kn + (n-k)n + (n-k)n + kn \right]$$

$$= 4n^2 \leq n(5n - 2),$$

it follows that

$$\xi^d(G) \geq \frac{1}{n(n-1)} \left[ n(5n - 2) - 2(k^2 + (n-k)n + (n-k)^2 + nk) \right]^2 + \frac{n(5n - 2)^2}{n-1}$$

$$= \frac{1}{n(n-1)} \left[ -4k^2 + 4nk + n(n-2) \right]^2 + \frac{n(5n - 2)^2}{n-1}.$$

Suppose $f(x) = -4x^2 + 4nx + n(n-2)$ with $1 \leq x \leq n - 1$. It is easy to see that $f_{\min}(x) = \min\{f(1), f(n-1)\}$. As $f(1) = f(n-1)$, $f_{\min}(x) = f(1) = n^2 + 2n - 4$, thus $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2 + 2n - 4)^2 + \frac{n(5n-2)^2}{n-1}$. 
If \( \xi^d(G) = \frac{1}{n(n-1)}(n^2 + 2n - 4)^2 + \frac{n(5n-2)^2}{n-1} \), then \( k = 1 \), \( d(x_1) = 1 \), \( d(x_2) = \cdots = d(x_n) = n \), \( d(y_1) = d(y_2) = \cdots = d(y_{n-1}) = n - 1 \), \( d(y_n) = n \). Thus \( G = K_{n,n} - K_{1,n-1} \), which is not Hamiltonian.

On the other hand, \( \varepsilon(v) \geq \frac{D(v)}{n-1} \), with equality if and only if \( d(v, u) \) (for fixed \( v \in V(G) \)) is a constant for all \( u \in V(G) \) with \( v \neq u \). However, \( G = K_{n,n} - K_{1,n-1} \) cannot satisfy it, the equality cannot hold.

(2) We consider \( \xi^{ce}(G) \).

\[
D(x_1) \geq d(x_1) + 2(n - 1) + 3(n - d(x_1)) = 5n - 2 - 2d(x_1).
\]

Similarly, for each \( i \) with \( 2 \leq i \leq n \) and each \( j \) with \( 1 \leq j \leq n \),

\[
D(x_i) \geq d(x_i) + 2(n - 1) + 3(n - d(x_i)) = 5n - 2 - 2d(x_i).
\]

\[
D(y_j) \geq d(y_j) + 2(n - 1) + 3(n - d(y_j)) = 5n - 2 - 2d(y_j).
\]

Therefore,

\[
\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} 
\leq \sum_{x_i \in X} \frac{n-1}{D(x_i)} \cdot d(x_i) + \sum_{y_j \in X} \frac{n-1}{D(y_j)} \cdot d(y_j) 
\leq (n-1) \left[ \sum_{x_i \in X} \frac{d(x_i)}{5n-2 - 2d(x_i)} + \sum_{y_j \in X} \frac{d(y_j)}{5n-2 - 2d(y_j)} \right].
\]

Suppose \( f(x) = \frac{x}{5n-2-2x} \), then we have \( f'(x) = \frac{5n-2-2x}{(5n-2-2x)^2} > 0 \), so

\[
\xi^{ce}(G) \leq (n-1) \left[ \frac{k^2}{5n-2 - 2k} + \frac{(n-k)n}{5n-2 - 2n} + \frac{(n-k)^2}{5n-2 - 2(n-k)} + \frac{kn}{5n-2 - 2n} \right].
\]

Since \( 1 \leq k \leq n-1 \), then,

\[
\xi^{ce}(G) \leq (n-1) \left[ \frac{k^2}{3k+3} + \frac{(n-k)^2}{3k+3} + \frac{n^2}{3n-2} \right].
\]

Suppose \( f(x) = \frac{x^2 + (n-x)^2}{3x+3} \) with \( 1 \leq x \leq (n-1) \). It is easy to see that \( f_{\max}(x) = \max\{ f(1), f(n-1) \} \).

As \( f(n-1) = \frac{1}{3n}(n^2 - 2n + 2) \), \( f(1) = \frac{1}{6}(n^2 - 2n + 2) \), \( f(n-1) - f(1) = -\frac{(n-2)(n^2 - 2n + 2)}{6n} < 0 \), so we have \( f_{\max}(x) = f(1) \).
If $\xi^c(G) = (n - 1) \left[ \frac{1}{6}(n^2 - 2n + 2) + \frac{n^2}{3n^2} \right]$, then $k = 1$, the remaining is as in the previous proof. $\square$

**Theorem 3.6.** Let $G$ be a 2-connected graph of order $n \geq 12$. If $\xi^d(G) \leq \frac{1}{n(n-1)} \left[ n^2 + 3n - 12 \right]^2$, then $G$ is Hamiltonian.

**Proof.** Suppose that $G$ is not Hamiltonian and $G$ is not $K_2 \lor (2K_1 \cup K_{n-4})$, then from Lemma 2.6, we have that $m \leq \left( \binom{n-2}{2} + 3 \right)$. As in Theorem 3.1,

$$\xi^d(G) \geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2.$$

Since $2n(n-1) - \sum_{v \in V(G)} d(v) > 0$, then

$$\xi^d(G) \geq \frac{1}{n(n-1)} \left[ 2n(n-1) - 2m \right]^2$$

$$\geq \frac{1}{n(n-1)} \left[ n^2 + 3n - 12 \right]^2.$$

On the other hand, $\varepsilon(v) \geq \frac{D(v)}{n-1}$, with equality if and only if $d(v, u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \neq u$. Thus, $K_2 \lor (2K_1 \cup K_{n-4})$ can not satisfy it, the equality can not hold. $\square$

**Theorem 3.7.** Let $G$ be a 3-connected graph of order $n \geq 18$. If $\xi^d(G) \leq \frac{1}{n(n-1)} \left[ n^2 + 5n - 28 \right]^2$, then $G$ is Hamiltonian.

**Proof.** Suppose that $G$ is not Hamiltonian and $G$ is not $K_3 \lor (3K_1 \cup K_{n-6})$. Then from Lemma 2.7, we have that $m \leq \left( \binom{n-3}{2} + 8 \right)$. Therefore,

$$\xi^d(G) \geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2$$

$$\geq \frac{1}{n(n-1)} \left[ 2n(n-1) - 2m \right]^2$$

$$\geq \frac{1}{n(n-1)} \left[ n^2 + 5n - 28 \right]^2.$$

On the other hand, $\varepsilon(v) \geq \frac{D(v)}{n-1}$, with equality if and only if $d(v, u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \neq u$. Thus, $K_3 \lor (3K_1 \cup K_{n-6})$ can not satisfy it, the equality can not hold. $\square$

**Theorem 3.8.** Let $G$ be a $k$-connected graph of order $n \geq 18$. If

$$\xi^d(G) \leq \frac{1}{n(n-1)} \left[ n(n-1) + (k+1)(n-k-1) \right]^2,$$

then $G$ is Hamiltonian.
\textbf{Proof.} Suppose that $G$ is not Hamiltonian, then from Lemma 2.8, we have that $m \leq \binom{n}{2} - (k + 1)(n - k - 1)/2$. Therefore,

$$\xi^d(G) \geq \frac{1}{n(n - 1)} \left( \sum_{v \in V(G)} d(v) - 2n(n - 1) \right)^2$$

$$= \frac{1}{n(n - 1)} [2n(n - 1) - 2m]^2$$

$$\geq \frac{1}{n(n - 1)} [n(n - 1) + (k + 1)(n - k - 1)]^2,$$

which is a contradiction. This completes the proof. \qed

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