ON NUMERICAL SEMIGROUPS WITH EMBEDDING DIMENSION THREE

ALI MAHDAVI AND FARHAD RAHMATI

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Abstract. Let $f \neq 1, 3$ be a positive integer. We prove that there exists a numerical semigroup $S$ with embedding dimension three such that $f$ is the Frobenius number of $S$. We also show that the same fact holds for affine semigroups in higher dimensional monoids.

1. Introduction and basic notations

This note is motivated by results of [8], that every positive integer is the Frobenius number of a numerical semigroup with at most three generators. The main result in this note is that every positive integer $f \neq 1, 3$, is the Frobenius number of a numerical semigroup with embedding dimension three (see Theorem 2.4). The set of nonnegative integers will be denoted by $\mathbb{N}$. A numerical semigroup $S$ is a submonoid of $\mathbb{N}$ such that $\mathbb{N}\setminus S$ is finite. Applications of numerical semigroups are found in the study of the parameters of Algebraic Geometry codes and cryptography (see [4],[6]). For a nonempty subset $A$ of $\mathbb{N}$, we will denote by $\langle A \rangle$ the submonoid of $\mathbb{N}$ generated by $A$. It is well known that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$ ([9, Lemma 2.1]). Let $S$ be a numerical semigroup generated by $A = \{a_1, a_2, \ldots, a_n\}$. If no proper subset of $A$ generates $S$, the set $A$ is called a minimal system of generators of $S$. Every numerical semigroup has a unique minimal system of generators which is a finite set ([9, Theorem 2.7]). The cardinality of the minimal system of generators of $S$ is called the embedding dimension of $S$ and will be denoted by $e(S)$. For $a \in S\setminus\{0\}$ we define the Apéry set of $a$ in $S$ as the set

$$\text{Ap}(S, a) = \{s \in S \mid s - a \notin S\}.$$
This set has precisely \( a \) elements, which can be denoted by \( \omega_0, \omega_1, \ldots, \omega_{a-1} \), where \( \omega_i \) is the smallest element of \( S \) in respective congruences class mod \( a \), for all \( i \in \{0, \ldots, a-1\} \) (see [9, Lemma 2.4]).

The Frobenius number of a numerical semigroup \( S \), generated by \( a_1, \ldots, a_n \) is the largest integer \( f^*(S) \) such that the linear equation \( a_1x_1 + \cdots + a_nx_n = f^*(S) \) does not have any non-negative integer solutions. Note that \( f^*(S) \) is the largest integer not belonging to \( S \). It is not hard to show that \( f^*(S) + a \) is the greatest element, \( \max(\text{Ap}(S, a)) \), in \( \text{Ap}(S, a) \). The Frobenius number of a semigroup has been investigated by several authors ([3],[5],[7]). For \( n = 2 \), Sylvester proved in [12] that \( f^*((a_1, a_2)) = a_1a_2 - a_1 - a_2 \). But if \( n \geq 3 \), no closed formula is known ([3]). From Sylvester's formula, for a two-generators numerical semigroup \( S \), \( f^*(S) \) is always an odd integer and for every odd integer \( f \), \( f^*((2, f + 2)) = f \).

The Frobenius problem is generalized to higher dimensional cases (see [1], [2], [13], [14]). In the last section we show that every vector \( f \in \mathbb{N}^n \setminus E_r \cup 3E_r \) is the minimal Frobenius vector of an affine semigroup \( S \) minimally generated by \( r + 1 \) elements.

2. One dimensional case

To prove our main result we start by proving the following proposition.

**Proposition 2.1.** Let \( f \) be a positive integer. If \( 12 | f \), then there exists a numerical semigroup \( S \) with \( e(S) = 3 \) such that \( f^*(S) = f \).

**Proof.** Let \( r \) and \( m \) be integers such that \( f = 3^r m \) with \( (3, m) = 1 \). Set \( a_1 = 3^{r+1}, a_2 = \frac{m}{2} + 3, a_3 = 3^r \frac{m}{4} + \frac{m}{2} + 3 \) and \( S = Na_1 + Na_2 + Na_3 \). Since \( (3, m) = 1 \), we have \( \gcd(a_1, a_2, a_3) = 1 \). We first show that \( e(S) = 3 \). Let \( a_3 = n_1a_1 + n_2a_2 \), for some \( n_1, n_2 \in \mathbb{N} \). As \( (3, m) = 1 \), we have \( n_2 \neq 0 \). So

\[
3^r \frac{m}{4} = n_13^{r+1} + (n_2 - 1)\left( \frac{m}{2} + 3 \right) \Rightarrow \left( \frac{m}{4} - 3n_1 \right)3^r = (n_2 - 1)\left( \frac{m}{2} + 3 \right)
\]

and thus \( 3^r \mid (n_2 - 1) \). Hence \( 3^r \leq n_2 - 1 \), which implies \( \frac{m}{2} + 3 \leq \frac{m}{4} - 3n_1 \), contradicting that \( m, n_1 \in \mathbb{N} \).

So \( e(S) = 3 \). One can easily check that

1. \( 3a_3 = \frac{m}{4}a_1 + 3a_2; \)
2. \( (\frac{a_1}{3} + 2)a_2 = a_1 + 2a_3; \)
3. \( (\frac{a_1}{3} - 1)a_2 + a_3 = (\frac{m}{4} + 1)a_1. \)

Every element in \( \text{Ap}(S, a_1) \) is a linear combination of \( a_2 \) and \( a_3 \). Let

\[
B = \{ 0, a_2, 2a_2, \ldots, (\frac{a_1}{2} + 1)a_2, a_3, 2a_3, a_2 + a_3, 2a_2 + a_3, \ldots, (\frac{a_1}{3} - 2)a_2 + 2a_3, 2a_2 + 2a_3, \ldots, (\frac{a_1}{3} - 2)a_2 + 3a_2 + a_3 \}. \]

Using the relations (1), (2) and (3), we have \( \text{Ap}(S, a_1) \subseteq B \) and since

\[
|B| = 1 + (\frac{a_1}{3} + 1) + 2 + (\frac{a_1}{3} - 2) + (\frac{a_1}{3} - 2) = a_1 = |\text{Ap}(S, a_1)|,
\]

the equality holds. Since \( 12 \mid f \), we have \( r \geq 1 \) and \( m \geq 4 \). So

\[
(\frac{a_1}{3} - 2)a_2 + 2a_3 - (\frac{a_1}{3} + 1)a_2 = 2a_3 - 3a_2 = 3^r \frac{m}{2} + m + 6 - 3^r \frac{m}{2} - 9 > 0.
\]
This implies that \( \max(\text{Ap}(S, a_1)) = (\frac{a_1}{3} - 2)a_2 + 2a_3 \). Hence

\[
f^*(S) = \max(\text{Ap}(S, a_1)) - a_1
= \left(\frac{a_1}{3} - 2\right)a_2 + 2a_3 - a_1
= \left(\frac{a_1}{3} - 2\right)a_2 + 2a_3 + a_1 - 2a_1
= \left(\frac{a_1}{3} - 2\right)a_2 + \left(\frac{a_1}{3} + 2\right)a_2 - 2a_1
= 3^r m.
\]

\[\square\]

By Lemma 1.1 and Proposition 1.2 in [8], we have the following proposition.

**Proposition 2.2.** Let \( f \) be a positive integer.

(1) If \( 3 \nmid f \), then \( f^*\langle (3, a, b) \rangle = f \), where \( \{a, b\} = \{x \in \{f + 1, f + 2, f + 3\} \mid 3 \nmid x\} \);

(2) If \( f \) is even and \( 4 \nmid f \), then \( f^*\langle (4, \frac{f}{2} + 2, \frac{f}{2} + 4) \rangle = f \);

(3) If \( (4, f) = 1 \), \( 3 \mid f \) and \( f > 12 \), then \( f^*\langle (4, \frac{f}{4} + 4, f + 4) \rangle = f \).

The following remark tells us that, when \( e(S) = 3 \), \( f^*(S) \neq 1, 3 \).

**Remark 2.3.** Let \( S \) be a numerical semigroup generated by \( A = \{a_1, a_2, a_3\} \) and let \( e(S) = 3 \).

(1) If \( f^*(S) = 1 \), then \( 2 \in A \) and thus \( e(S) = 2 \), a contradiction;

(2) Let \( f^*(S) = 3 \). Since \( e(S) = 3 \), so \( 1, 2 \notin A \). Thus \( A = \{4, 5, 6\} \) and \( f^*(S) = 7 \), a contradiction.

Our main result is the following theorem.

**Theorem 2.4.** Let \( f \neq 1, 3 \) be a positive integer. Then there exists a numerical semigroup \( S \) with \( e(S) = 3 \) such that \( f^*(S) = f \).

**Proof.** Every positive integer can be written as \( 12n + i \) for \( i \in \{0, 1, 2, \ldots, 11\} \) by the division algorithm. To prove the theorem, we consider five cases.

**case 1:** If \( 3 \nmid f \), the claim follows by Proposition 2.2.

**case 2:** If \( f = 12n, n > 0 \), the claim follows by Proposition 2.1.

**case 3:** If \( f = 12n + 3, n > 0 \), we set \( S = \langle 4, 4n + 5, 12n + 7 \rangle \). From Proposition 2.2, \( f^*(S) = 12n + 3 \). We show that \( e(S) = 3 \). Since \( 12n + 7 \) has a remainder of 3 modulo 4, \( 4n + 5 \) has a remainder of 1 modulo 4 and \( 2(4n + 5) \) has a remainder of 2 modulo 4, we conclude that \( 12n + 7 \notin \langle 4, 4n + 5 \rangle \). So \( e(S) = 3 \).

**case 4:** If \( f = 12n + 6, n \geq 0 \), we set \( S = \langle 4, 6n + 5, 6n + 7 \rangle \). Clearly \( 6n + 7 \notin \langle 4, 6n + 5 \rangle \). So \( e(S) = 3 \). Moreover from Proposition 2.2, \( f^*(S) = 12n + 6 \).

**case 5:** \( f = 12n + 9, n \geq 0 \). There are two cases to consider.

(1) \( f = 9 \). It is easy to see that the Frobenius number of \( S = \langle 5, 6, 8 \rangle \) is \( f = 9 \).

(2) If \( f = 12n + 9, n \geq 1 \), we set \( S = \langle 4, 4n + 7, 12n + 13 \rangle \). From Proposition 2.2, \( f^*(S) = 12n + 9 \). We show that \( e(S) = 3 \). Since \( 12n + 13 \) has a remainder of 1 modulo 4, \( 4n + 7 \) has a remainder of 3
modulo 4 and $2(4n + 7)$ has a remainder of 2 modulo 4, we conclude that $12n + 13 \not\in \langle 4, 4n + 7 \rangle$. So $e(S) = 3$. □

**Remark 2.5.** From Theorem 2.4, the minimal generators of the numerical semigroup $S$, associated to the given Frobenius number $f$, are constructed explicitly.

### 3. Higher dimensional case

Let $A = \{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ be a subset of $\mathbb{N}^r$ for some positive integer $r \geq 2$ and let

$$S = \mathbb{N}A = \left\{ \sum_{i=1}^{n} n_i v_i \mid n_i \in \mathbb{N} \right\}$$

be the affine semigroup generated by $A$. The group spanned by $S$, denoted by $G$, is defined as $G = \{u - v \mid u, v \in S\}$. We can define the following relation on $G$: for any $u, v \in G$, $v \preceq u$ if $u - v \in S$. The cone spanned by $S$ and interior of the cone spanned by $S$, are denoted by:

$$C = \left\{ \sum_{i=1}^{n} q_i v_i \mid q_i \in \mathbb{Q}_{\geq 0} \right\} \quad \text{and} \quad C^o = \left\{ \sum_{i=1}^{n} q_i v_i \mid q_i \in \mathbb{Q}_{> 0} \right\}$$

respectively. The vector $f^* \in G \setminus S$ is called a *Frobenius vector* for $S$, if for all $x \in C^o \cap G$, we have $f^* + x \in S$ (see Fig 1). Moreover, the Frobenius vector $f^*$ is called minimal, if there is no Frobenius vector $f$ such that $f^* \in f + C$.

![Fig.1](image)

**Definition 3.1.** The semigroup $S$ is called simplicial if there exist $v_{i_1}, v_{i_2}, \ldots, v_{i_r} \in A$ such that $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ are linearly independent over $\mathbb{Q}$ and $C = \mathbb{Q}_{\geq 0} v_{i_1} + \cdots + \mathbb{Q}_{\geq 0} v_{i_r}$.

If $r$ is less than three, every affine semigroup is simplicial. Assume without loss of generality that $\{i_1, \ldots, i_r\} = \{1, \ldots, r\}$. For simplicial affine semigroups, the set $T = \cap_{i=1}^{r} (\text{Ap}(S, v_i))$ is always finite (see [10, Section 1]). By a free semigroup we mean the following (for more details, please see [11]).

**Definition 3.2.** The semigroup $S$ is called free, if the cardinality of $T$ is $c_{r+1} c_{r+2} \cdots c_n$, where

$$c_i = \min\{k \in \mathbb{N}_{> 0} \mid k v_i \in \langle v_1, \ldots, v_{i-1} \rangle\}, i = r + 1, \ldots, n.$$ 

In [2], the author proves the following proposition.
Proposition 3.3. With the above notation, assume that $S$ is free and $\eta = \max \leq T$. Then $S$ has a unique minimal Frobenius vector of the form $f^*(S) = \eta - \sum_{i=1}^r v_i$.

Let $E_r = \{e_1, \ldots, e_r\}$ be the standard basis of $\mathbb{N}^r$ and $3E_r = \{3e_1, \ldots, 3e_r\}$. The following is our main result.

Theorem 3.4. Let $f$ be a nonzero vector in $\mathbb{N}^r$, $r \geq 2$, and $f \notin E_r \cup 3E_r$. Then there exists an affine semigroup $S \subset \mathbb{N}^r$ generated by at most $r + 1$ elements, such that $f$ is the unique minimal Frobenius vector of $S$.

Proof. Let $f = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $f$ has at least two nonzero components. For every $i = 1, \ldots, r$, if $\alpha_i \neq 0$, we set $v_i = (0, \ldots, 0, 3\alpha_i, 0, \ldots, 0)$. Let $\alpha_{i_1}, \alpha_{i_2}, \ldots$ and $\alpha_{i_r}$ be nonzero components of $f$. By assumption $t > 1$. We set $S = (v_{i_1}, \ldots, v_{i_r}, 2f)$. Since $2f = \frac{2}{3}v_{i_1} + \cdots + \frac{2}{3}v_{i_r}$, without loss of generality we can assume that $S$ is a simplicial affine semigroup in $\mathbb{N}^r$. It is not hard to see that $\min \{k \in \mathbb{N} \mid k(2f) \in Nv_{i_1} + \cdots + Nv_{i_r}\}$ is equal to 3 and $T = \{0, 2f, 4f\}$. So $S$ is free and $\eta = 4f$.

By using Proposition 3.3, we conclude that $f^*(S) = \eta - (v_{i_1} + \cdots + v_{i_r}) = f$.

Now let $f$ has only one nonzero component. Let $\alpha_k \neq 0, k \in \{1, \ldots, r\}$. By using Theorem 2.4, there exists a numerical semigroup $S_k = (a_1, a_2, a_3)$ with $e(S_k) = 3$, such that $f^*(S_k) = \alpha_k$. Let $S$ be the affine semigroup generated by $\{v_1 = a_1e_k, v_2 = a_2e_k, v_3 = a_3e_k\}$. It is not hard to see that $f$ is the minimal Frobenius vector of $S$. \hfill $\square$

Corollary 3.5. Let $f$ be a nonzero vector in $\mathbb{N}^2$, and $f \notin E_2 \cup 3E_2$. Then there exists an affine semigroup $S \subset \mathbb{N}^2$ minimally generated by three elements, such that $f$ is the unique minimal Frobenius vector of $S$.

Proof. There are two cases. (i) $f$ has two nonzero components, and (ii) $f$ has one nonzero component. In both cases there exists an affine semigroup minimally generated by three elements, such that $f^*(S) = f$. \hfill $\square$

Example 3.6. Let $f = (12, 23)$. We set $v_1 = (36, 0), v_2 = (0, 69), v_3 = (24, 46)$ and $S = Nv_1 + Nv_2 + Nv_3$. So $f^*(S) = 2v_3 - v_1 - v_2 = (12, 23)$ (see Fig 2).
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REFERENCES


Ali Mahdavi
Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology
Tehran, Iran
Email: a.mahdavi@aut.ac.ir

Farhad Rahmati
Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology
Tehran, Iran
Email: frahmati@aut.ac.ir