A NOTE ON FINITE C-TIDY GROUPS

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Abstract. Let $G$ be a group and $x \in G$. The cyclicizer of $x$ is defined to be the subset $\text{Cyc}(x) = \{ y \in G \mid \langle x, y \rangle \text{ is cyclic} \}$. $G$ is said to be a tidy group if $\text{Cyc}(x)$ is a subgroup for all $x \in G$. We call $G$ to be a C-tidy group if $\text{Cyc}(x)$ is a cyclic subgroup for all $x \in G \setminus K(G)$, where $K(G)$ is the intersection of all the cyclicizers in $G$. In this note, we classify finite C-tidy groups with $K(G) = \{1\}$.

1. Introduction

Let $G$ be a group and $x \in G$. The centralizer of $x$, denoted by $C_G(x)$, is defined by $C_G(x) = \{ y \in G \mid yx = xy \}$. The cyclicizer of $x$, denoted by $\text{Cyc}_G(x)$, is defined to be the subset $\text{Cyc}_G(x) = \{ y \in G \mid \langle x, y \rangle \text{ is cyclic} \}$. If the context is clear we will write $C(x)$ and $\text{Cyc}(x)$ in place of $C_G(x)$ and $\text{Cyc}_G(x)$ respectively. In general, $\text{Cyc}(x)$ is not a subgroup. For example, in the group $C_2 \times C_4$, we have

$$\text{Cyc}((0, 2)) = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 3)\},$$

which is not a subgroup of $C_2 \times C_4$. Following [15], a group $G$ is said to be a tidy group if $\text{Cyc}(x)$ is a subgroup for all $x \in G$. We call $G$ to be a C-tidy group if $\text{Cyc}(x)$ is a cyclic subgroup for all $x \in G \setminus K(G)$, where $K(G)$ is the intersection of all the cyclicizers in $G$.

Starting with D. Patrick and E. Wepsic in 1991, [16] many authors have studied and characterised groups (finite and infinite) in terms of cyclicizers and tidy properties (see [15], [2], [3], [6], [14]). In this paper, we continue with this problem and classify finite C-tidy groups $G$ with $K(G) = \{1\}$.

Throughout this paper, all groups are finite and all notations are usual. For example, $C_n$ denotes the cyclic group of order $n$ and $C_n^k$ denotes $C_n \times \cdots \times C_n$, $k$-times, where $n$ and $k$ are positive integers.

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PGL($n, q$) and PSL($n, q$) denote the projective general linear and the projective special linear group of degree $n$ over the field of size $q$ respectively, $Sz(2^{2m+1}), m > 0$ denotes the Suzuki group over the field with $2^{2m+1}$ elements. For a group $G$, 1 and $Z(G)$ denote the identity element and the center respectively.

2. Some basic results

We begin with some definitions concerning coverings and partitions of a group. It may be mentioned here that the first paper concerning partitions of groups was published in 1906 by G. A. Miller [13].

**Definition:** A collection $\Pi$ of non-trivial subgroups of a group $G$ is called a covering if every element of $G$ belongs to a subgroup in $\Pi$. The covering $\Pi$ is called a partition if every non-trivial element of $G$ belongs to a unique subgroup in $\Pi$. If $|\Pi| = 1$, the partition is said to be trivial. The subgroups in $\Pi$ are called the components of $\Pi$. A partition $\Pi$ of a group $G$ is said to be normal if $g^{-1}Xg \in \Pi$ for every $X \in \Pi$ and $g \in G$. Following Kontorovich, a group $G$ is said to be completely decomposable if it has a partition $\Pi$ such that every component of $\Pi$ is cyclic. See ([19]) for more information in this regard.

**Proposition 2.1.** Let $G$ be a C-tidy group. Then for any $x, y \in G \setminus K(G)$ either $Cyc(x) = Cyc(y)$ or $Cyc(x) \cap Cyc(y) = K(G)$.

**Proof.** Suppose $Cyc(x) \neq Cyc(y)$. Let $a \in Cyc(x) \cap Cyc(y) \setminus K(G)$. Then $a \in Cyc(x)$ and since $Cyc(x)$ is cyclic, therefore $Cyc(x) \subseteq Cyc(a)$. Similarly, $Cyc(a) \subseteq Cyc(x)$ and hence $Cyc(a) = Cyc(x)$. In the same way, we can show that $Cyc(a) = Cyc(y)$, which is a contradiction. □

We now prove the following proposition which will be used later.

**Proposition 2.2.** Let $G$ be a group with a partition $\Pi$. Then $G$ is tidy (C-tidy) if and only if every component is tidy (C-tidy).

**Proof.** Let $\Pi = \{X_i\}_{i \in I}$ and $x \in G \setminus \{1\}$. Then $x \in X_i$ for some $i \in I$. Suppose $y \in Cyc_G(x) \setminus \{1\}$. Then $\langle y, x \rangle = \langle t \rangle$ for some $t \in G$. Hence $x = t^k$ for some positive integer $k$. Suppose $t \notin X_i$. Then $t \in X_j$ for some $j \neq i$. Therefore $t^k \in X_i \cap X_j$, which is a contradiction. Hence $y \in X_i$ and so $Cyc_G(x) \subseteq X_i$. Now, since $Cyc_{X_i}(x) = Cyc_G(x) \cap X_i = Cyc_G(x)$ for any $x \in X_i \setminus \{1\}$ and $i \in I$, therefore the result follows. □

As a consequence of Proposition 2.1 and Proposition 2.2, we get the following equivalent condition for completely decomposable groups.

**Proposition 2.3.** Let $G$ be a non-cyclic group. Then $G$ is C-tidy with $K(G) = \{1\}$ if and only if $G$ is completely decomposable.
Proof. Let $G$ be C-tidy with $K(G) = \{1\}$. Let $\Pi$ be the collection of all proper cyclicizers of $G$. By Proposition 2.1, $\Pi$ is a partition of $G$ such that every component of $\Pi$ is cyclic. Hence $G$ is completely decomposable.

Conversely, suppose $G$ is completely decomposable. Then by Proposition 2.2, $G$ is C-tidy with $K(G) = \{1\}$. □

A group $G$ is said to be a Frobenius group if it contains a subgroup $H$ such that $\{1\} \neq H \neq G$ and $H \cap H^g = \{1\}$ for all $g \in G \setminus H$. A subgroup with these properties is called a Frobenius complement of $G$. The Frobenius kernel of $G$, with respect to $H$, is defined by $K = \langle G \setminus \bigcup_{g \in G} H^g \rangle \cup \{1\}$. It is well known that (see [17]) $K$ is nilpotent and $G$ has a partition $\{K, H^{g_1}, H^{g_2}, \ldots, H^{g_m}\}$ with $K = \{g_1, g_2, \ldots, g_m\}$, the so called Frobenius partition. By Problem 7.1 of [10], it follows that $H^{g_i}, 1 \leq i \leq m$ are Frobenius complements of $G$.

The following simple lemma will be used in proving the next proposition.

Lemma 2.4. Let $G$ be a Frobenius group with Frobenius complement $H$. Then $H$ is a tidy group.

Proof. By Lemma 3, p. 273 of [5], every sylow subgroup of $H$ is either cyclic or generalized quaternion. Therefore by Theorem 6 of [10], $\text{Cyc}_H(x) = \text{C}_H(x)$ for all $x \in H$. □

Proposition 2.5. Let $G$ be a Frobenius group with Frobenius kernel $K$. Then $G$ is tidy if and only if $K$ is tidy.

Proof. If $K$ is tidy, then by Proposition 2.4, every component of the Frobenius partition of $G$ is tidy and by Proposition 2.2, $G$ is tidy. Converse is trivial. □

In the following theorem, $I(G)$ denotes the set of all solutions of the equation $x^2 = 1$ in $G$.

Theorem 2.6 ((See [6], Theorem 2.4)). The Suzuki group $Sz(2^{2m+1}), m > 0$ is not tidy.

Proof. Let $G = Sz(2^{2m+1}), m > 0$ and $S$ be a sylow 2-subgroup of $G$. By p. 12 of [7], $|Z(S)| = 2^{2m+1}, Z(S) = I(G)$ and $S$ is of exponent 4. Therefore by Theorem 14 of [15], $S$ is not tidy and hence $G$ is not tidy. □

Let $G$ be a group and $p$ be a prime. We recall that the Hughes subgroup denoted by $H_p(G)$ is defined to be the subgroup generated by all the elements of $G$ whose order is not $p$. The group $G$ is said to be a group of Hughes-Thompson type if it is not a $p$-group and $H_p(G) \neq G$ for some prime $p$.

The following theorem gives a classification of all groups with a non-trivial partition (see [19]).

Theorem 2.7. A group $G$ has a non-trivial partition if and only if it satisfies one of the following conditions:

1. $G$ is a $p$-group with $H_p(G) \neq G$ and $|G| > p$, $p$ being a prime,
2. $G$ is a Frobenius group,
(3) G is a group of Hughes-Thompson type,
(4) G is isomorphic with $\text{PGL}(2, p^h)$, $p$ being an odd prime, $h \geq 1$,
(5) G is isomorphic with $\text{PSL}(2, p^h)$, $p$ being a prime, $h \geq 1$,
(6) G is isomorphic with a Suzuki group $S\text{z}(2^{2m+1}), m > 0$.

3. C-tidy groups with $K(G) = \{1\}$

In this section we study the groups in which $\text{Cyc}(x)$ is a cyclic subgroup for all $x \in G \setminus \{1\}$. We begin with the following definition.

Definition: A group G is called a C-tidy group if $\text{Cyc}(x)$ is a cyclic subgroup for all $x \in G \setminus K(G)$.

Note that there exists tidy groups which are not C-tidy groups. For example, the alternating group $A_7$ is tidy (see Theorem 1.3 of [1]), but not C-tidy, because $(4 \, 5)(6 \, 7), (4 \, 6)(5 \, 7) \in \text{Cyc}((1 \, 2 \, 3))$.

We now classify all nilpotent C-tidy groups with $K(G) = \{1\}$.

**Theorem 3.1.** G is a nilpotent C-tidy group with $K(G) = \{1\}$ if and only if G is a non-cyclic $p$-group with a cyclic normal subgroup H such that $o(x) = p$ for all $x \in G \setminus H$, where $p$ is a prime.

**Proof.** Clearly G cannot be cyclic and we have $G = P \times A$, where $P$ is a non-cyclic $p$-group and $A$ is a group with $\gcd(p, |A|) = 1$, $p$ being a prime. By Lemma 2.12 of [14], $K(P) = K(A) = \{1\}$. Suppose $A \neq \{1\}$. Let $a \in P \setminus K(P)$ and $b \in K(A)$. Then by Lemma 10 of [15], we have $\text{Cyc}((a, b)) = \text{Cyc}(a) \times A$, which is not cyclic, a contradiction. Hence $A = \{1\}$. Therefore G is a $p$-group. By Theorem 14 of [15], there exists $H \subseteq G$ such that H is cyclic or generalized quaternion and for all $x \in G \setminus H$ we have $o(x) = p$. If H is generalized quaternion, then by Theorem 8 of [15], $K(H) \neq \{1\}$. Let $y \in K(H) \setminus \{1\}$. Then $H \subseteq \text{Cyc}(y)$, which is a contradiction since $\text{Cyc}(y)$ is cyclic. Hence H is cyclic.

Conversely, by Theorem 8 of [15], we have $K(G) = \{1\}$. Suppose $h \in H \setminus \{1\}$. Clearly $H \subseteq \text{Cyc}(h)$. Suppose $H \not\subseteq \text{Cyc}(h)$. Let $g \in \text{Cyc}(h) \setminus H$. Then $\langle g, h \rangle$ is cyclic and so $\langle g, h \rangle$ has exactly one subgroup of order $p$, namely $\langle g \rangle$. But $p \mid |\langle h \rangle|$ and so $\langle h \rangle$ has a subgroup of order $p$. Hence $\langle g \rangle \subseteq \langle h \rangle \subseteq H$, which is a contradiction. Hence $H = \text{Cyc}(h)$.

Again, since $o(x) = p$ for all $x \in G \setminus H$, therefore $\text{Cyc}(x) = \langle x \rangle$ for all $x \in G \setminus H$. Hence G is a C-tidy group with $K(G) = \{1\}$.

**Corollary 3.2.** G is an abelian C-tidy group with $K(G) = \{1\}$ if and only if $G \cong C_p^k$ for some prime $p$ and some integer $k > 1$.

**Proof.** By Theorem 3.1, G is a p-group, where $p$ is a prime and by Theorem 11 of [15], we have $G \cong C_p^k$ for some integer $k > 1$. Converse is trivial.

Using Proposition 2.2, we now prove the following theorem.
Proof. We have $\text{PSL}(2, 2) \cong \text{PGL}(2, 2) \cong S_3$, $\text{PSL}(2, 3) \cong A_4$ and $\text{PSL}(2, 3) \cong S_4$ and one can easily observe that $S_3, A_4$ and $S_4$ are $C$-tidy groups with $K(S_3) = K(A_4) = K(S_4) = \{1\}$. Suppose $p^h \geq 4$. If $G = \text{PSL}(2, p^h)$ or $G = \text{PGL}(2, p^h)$, then by Proposition 2.4 of \[1\] and Proposition 3.21 of \[1\], $G$ contains subgroups $P, A$ and $B$ such that $P$ is elementary abelian $p$ group, $A, B$ are cyclic and $\Pi = \{P^x, A^x, B^x| x \in G\}$ is a partition for $G$. Again, we have $\text{Cyc}_{P^x}(y) = \langle y \rangle$ for any $y \in P^x \setminus \{1\}$. Hence by Proposition 2.2, $G$ is $C$-tidy and it is easy to see that $K(G) = \{1\}$. 

For a group $G$ and any $g, x \in G$, it is an easy exercise to show that $C(gxg^{-1}) = gC(x)g^{-1}$. The following proposition is an analog to this result.

**Proposition 3.4.** Let $G$ be a group. Then $\text{Cyc}(gxg^{-1}) = g\text{Cyc}(x)g^{-1}$ for any $g, x \in G$.

*Proof.* Suppose $a \in \text{Cyc}(x)$. Then $\langle a, x \rangle$ is cyclic and so $g\langle a, x \rangle g^{-1}$ is cyclic. But $g\langle a, x \rangle g^{-1} = \langle gag^{-1}, gxg^{-1} \rangle$. Hence $gag^{-1} \in \text{Cyc}(gxg^{-1})$.

Conversely, suppose $a \in \text{Cyc}(gxg^{-1})$. Then $\langle a, gxg^{-1} \rangle$ is cyclic. Therefore $\langle g^{-1}ag, x \rangle$ is cyclic. Hence $g^{-1}ag \in \text{Cyc}(x)$ and so $a \in g\text{Cyc}(x)g^{-1}$.

**Theorem 3.5.** Let $G$ be a $C$-tidy group with $K(G) = \{1\}$. If $G$ is non-solvable, then $G \cong \text{PGL}(2, p^h)$, $p$ odd or $G \cong \text{PSL}(2, p^h)$ for some prime $p$ and $h \geq 1$.

*Proof.* Let $\Pi$ be the collection of all proper cyclicizers of $G$. By Proposition 2.1 and Proposition 3.4, $\Pi$ is a normal non-trivial partition of $G$ in which all components are nilpotent. Therefore by Suzuki [18], [19], we have $G \cong \text{PGL}(2, p^h)$, $p$ odd or $G \cong \text{PSL}(2, p^h)$ for some prime $p$ and $h \geq 1$ or the Suzuki group $Sz(2^{2m+1}), m > 0$. But by Theorem 2.6, $Sz(2^{2m+1}), m > 0$ is not tidy. Hence $G \cong \text{PGL}(2, p^h)$, $p$ odd or $G \cong \text{PSL}(2, p^h)$ for some prime $p$ and $h \geq 1$.

The following theorem classifies all C-tidy groups with $K(G) = \{1\}$ in which the proper cyclicizers have equal order.

**Theorem 3.6.** Let $G$ be a group. Then $G$ is $C$-tidy, $K(G) = \{1\}$ and $|\text{Cyc}(x)| = |\text{Cyc}(y)|$ for all $x, y \in G \setminus \{1\}$ if and only if $G$ is a non-cyclic $p$-group of exponent $p$, for some prime $p$.

*Proof.* Let $\Pi$ be the collection of all proper cyclicizers of $G$. By Proposition 2.1, $\Pi$ is a non-trivial partition of $G$ in which all components of $\Pi$ have same order. Therefore by \[9\], $G$ is a $p$-group of exponent $p$, for some prime $p$. Again, since $K(G) = \{1\}$, therefore $G$ is non-cyclic.

Conversely, suppose $G$ is a non-cyclic $p$-group of exponent $p$, for some prime $p$. Then $\text{Cyc}(x) = \langle x \rangle$ for any $x \in G \setminus \{1\}$ and so $|\text{Cyc}(x)| = |\text{Cyc}(y)|$ for all $x, y \in G \setminus \{1\}$ and $K(G) = \{1\}$.

**Theorem 3.7.** Let $G$ be a $C$-tidy group with $K(G) = \{1\}$. If $\text{Cyc}(x)$ is subnormal for all $x \in G \setminus \{1\}$, then $G$ is a $p$-group for some prime $p$. 

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*Note:* The theorems and propositions are extracted from the text and presented in a natural format. The proofs are provided as they appear in the document. The theorems are numbered as in the original text. The propositions are numbered for clear reference. The document itself is cited as necessary. The text is formatted to maintain the structure and flow of the original document.
Proof. Let II be the collection of all proper cyclicizers of $G$. By Proposition 2.1, II is a non-trivial partition of $G$. Now, the result follows from a result of Kegel (see p. 574 of [19], [12]). □

**Theorem 3.8.** Let $G$ be a group. Then $G$ is C-tidy, $K(G) = \{1\}$ and $\text{Cyc}(x) \subseteq G$ for all $x \in G \setminus \{1\}$ if and only if $G \cong C_p^k$ for some prime $p$ and some integer $k > 1$.

**Proof.** Let II be the collection of all proper cyclicizers of $G$. By Proposition 2.1, II is a non-trivial partition of $G$ such that $\text{Cyc}(x)\text{Cyc}(y) = \text{Cyc}(y)\text{Cyc}(x)$ for all $x, y \in G \setminus \{1\}$. Therefore by Theorem 8 of [9], $G \cong C_p^k$ for some prime $p$ and some integer $k > 1$. Converse is trivial. □

It is easy to see that if $G$ is a C-tidy group then any proper subgroup of $G$ is also C-tidy. For C-tidy groups with trivial center, we have the following result.

**Theorem 3.9.** Let $G$ be a group with trivial center. Then $G$ is C-tidy if and only if $H$ is cyclic or C-tidy with $K(H) = \{1\}$ for any $H \leq G$.

**Proof.** Suppose $H$ is cyclic or C-tidy with $K(H) = \{1\}$ for any $H \leq G$. Then by Proposition 2.3, $H$ is completely decomposable. Hence by Kontorovich (p. 573 of [19]), $G$ is completely decomposable and by Proposition 2.3, $G$ is C-tidy.

Conversely, suppose $H$ is a non-cyclic proper subgroup of $G$. Clearly $H$ is C-tidy. If $x \in K(H) \setminus \{1\}$, then $H \subseteq \text{Cyc}(x)$, which forces $H$ to be cyclic, a contradiction. □

The following theorem classifies all C-tidy groups $G$ with $K(G) = \{1\}$ having non-trivial center. Recall that the fitting subgroup $F(G)$ of a group $G$ is the product of all nilpotent normal subgroups of $G$ and it is the largest nilpotent normal subgroup of $G$.

**Theorem 3.10.** Let $G$ be a group with non-trivial center. Then $G$ is C-tidy with $K(G) = \{1\}$ if and only if it satisfies one of the following conditions:

1. $G$ is a non-cyclic $p$-group with a cyclic normal subgroup $H$ such that $o(x) = p$ for all $x \in G \setminus H$, where $p$ is a prime,
2. $G$ is a non- $p$-group with a cyclic normal subgroup $H$ of prime index such that $|\text{Cyc}(x)| = p$ for all $x \in G \setminus H$, where $p$ is a prime.

**Proof.** If $G$ is a $p$-group for some prime $p$, then (1) follows from Theorem 3.1. Suppose $G$ is not a $p$-group, where $p$ is a prime. Let II be the collection of all proper cyclicizers of $G$. By Proposition 2.1, II is a non-trivial partition of $G$. Now, by a result of Kegel [12], the fitting subgroup $F(G)$ is cyclic normal of prime index $p$. Therefore $F(G) \subseteq \text{Cyc}(y) \subseteq G$ for any $y \in F(G) \setminus K(G)$ and so $F(G) = \text{Cyc}(y)$. Let $x \in G \setminus F(G)$. By Proposition 2.1, $\text{Cyc}(x) \cap \text{Cyc}(y) = K(G) = \{1\}$. Now, $G = \text{Cyc}(x)\text{Cyc}(y)$ and so $|G| = |\text{Cyc}(x)||\text{Cyc}(y)|$. Therefore $|\text{Cyc}(x)| = p$. Hence (2) follows.

Conversely, suppose the condition (1) holds. Then by Theorem 3.1, $G$ is C-tidy with $K(G) = \{1\}$. Next, suppose $H$ is a cyclic normal subgroup of $G$ of prime index and $|\text{Cyc}(x)| = p$ for all $x \in G \setminus H$, where $p$ is a prime. Then $H \subseteq \text{Cyc}(y) \subseteq G$ for any $y \in H \setminus K(G)$ and so $\text{Cyc}(y) = H$. Hence $G$ is C-tidy with $K(G) = \{1\}$. □
We now classify Frobenius C-tidy groups.

**Theorem 3.11.** Let $G$ be a Frobenius group with Frobenius kernel $K$ and $p$ be a prime. Then $G$ is C-tidy if and only if $K$ is cyclic or $K$ is a $p$-group with a cyclic normal subgroup $N$ such that $o(x) = p$ for all $x \in K \setminus N$ and $C(y)$ is cyclic for all $y \in G \setminus K$.

**Proof.** Suppose $G$ is a C-tidy Frobenius group with Frobenius kernel $K$. Then $K$ is a nilpotent C-tidy group. Suppose $K$ is not cyclic. Then $K(K) = \{1\}$, and by Theorem 3.1, $K$ is a $p$-group with a cyclic normal subgroup $N$ such that $o(x) = p$ for all $x \in K \setminus N$, $p$ being a prime.

Next, suppose $y \in G \setminus K$. Then $y \in H^9$ for some $g \in K$ where $H^9$ is a component of the Frobenius partition. Note that $H^9$ is a Frobenius complement and so by Lemma 3, p. 273 of [5], every sylow subgroup of $H^9$ is either cyclic or generalized quaternion. Therefore by Theorem 6 of [16] and Problem 7.1 of [10], $\text{Cyc}_{H^9}(y) = C_{H^9}(y) = C_G(y) = C(y)$. Hence $C(y)$ is cyclic. Converse follows from Theorem 3.1.

The following theorem classifies Hughes-Thompson type C-tidy groups with $K(G) = \{1\}$.

**Theorem 3.12.** Let $G$ be a group of Hughes-Thompson type. Then $G$ is C-tidy with $K(G) = \{1\}$ if and only if $H_p(G)$ is cyclic normal or $H_p(G)$ is a normal $q$-group with a cyclic normal subgroup $N$ such that $o(h) = q$ for all $h \in H_p(G) \setminus N$ and $|\text{Cyc}(x)| = p$ for all $x \in G \setminus H_p(G)$, where $p,q$ are distinct primes.

**Proof.** We have $H_p(G) \neq G$ for some prime $p$. By a theorem of Hughes-Thompson [3], $|G : H_p(G)| = p$ and by a theorem of Kegel [11], $H_p(G)$ is nilpotent.

Now, suppose $H_p(G)$ is cyclic. Then $H_p(G) = \langle a \rangle$ for some $a \in H_p(G)$. Let $g \in G$. Then $gH_p(G)g^{-1} = g\langle a \rangle g^{-1} = \langle gag^{-1} \rangle$. Therefore $o(gag^{-1}) \neq p$ and so $gag^{-1} \in H_p(G)$. Hence $gH_p(G)g^{-1} = H_p(G)$ and $H_p(G) \not\leq G$. Again, $H_p(G) = \text{Cyc}(y)$ for any $y \in H_p(G) \setminus \{1\}$. Therefore for any $x \in G \setminus H_p(G)$, we have $\text{Cyc}(x) = \langle x \rangle$ and so $|\text{Cyc}(x)| = p$.

Next, suppose $H_p(G)$ is not cyclic. Then $K(H_p(G)) = \{1\}$ and by Theorem 3.1, $H_p(G)$ is a $q$-group for some prime $q$ with a cyclic normal subgroup $N$ such that $o(h) = q$ for all $h \in H_p(G) \setminus N$. Again, if $q = p$, then $G$ will be a $p$-group, which is a contradiction. Hence $q \neq p$. Clearly $H_p(G)$ is the sylow $q$-subgroup of $G$ and so $H_p(G) \not\leq G$. Now, suppose $x \in G \setminus H_p(G)$ and $b \in \text{Cyc}(x)$. If $b \in H_p(G) \setminus \{1\}$, then $o(bx) \neq p$ and so $bx \in H_p(G)$. Therefore $x = b^{-1}bx \in H_p(G)$, which is a contradiction. Hence $\text{Cyc}(x) = \langle x \rangle$ and so $|\text{Cyc}(x)| = p$.

Conversely, suppose $H_p(G)$ is cyclic and $|\text{Cyc}(x)| = p$ for all $x \in G \setminus H_p(G)$ where $p$ is a prime. Then $G$ is C-tidy with $K(G) = \{1\}$. Next, suppose $H_p(G)$ is a $q$-group with a cyclic normal subgroup $N$ such that $o(h) = q$ for all $h \in H_p(G) \setminus N$ and $|\text{Cyc}(x)| = p$ for all $x \in G \setminus H_p(G)$, where $p,q$ are distinct primes. Let $z \in H_p(G) \setminus \{1\}$. Then $\text{Cyc}_{G}(z) = \text{Cyc}_{H_p(G)}(z)$. Now, the result follows by Theorem 3.1.

Finally, combining Proposition 2.1, Theorem 2.6, Theorem 2.7, Theorem 3.1, Theorem 3.3, Theorem 3.5, Theorem 3.11 and Theorem 3.12 we get the classification theorem for C-tidy groups with $K(G) = \{1\}$ as follows.
Theorem 3.13. A group $G$ is C-tidy with $K(G) = \{1\}$ if and only if it satisfies one of the following conditions:

1. $G$ is a non-cyclic $p$-group with a cyclic normal subgroup $H$ such that $o(x) = p$ for all $x \in G \setminus H$, $p$ being a prime,
2. $G$ is a Frobenius group in which the Frobenius kernel $K$ is cyclic or $K$ is a $p$-group with a cyclic normal subgroup $N$ such that $o(x) = p$ for all $x \in K \setminus N$ and $C(y)$ is cyclic for all $y \in G \setminus K$, $p$ being a prime,
3. $G$ is of Hughes-Thompson type in which $H_p(G)$ is cyclic normal or $H_p(G)$ is a normal $q$-group with a cyclic normal subgroup $N$ such that $o(h) = q$ for all $h \in H_p(G) \setminus N$ and $|Cyc(x)| = p$ for all $x \in G \setminus H_p(G)$, $p, q$ being distinct primes,
4. $G \cong \text{PGL}(2, p^h)$, $p$ being an odd prime, $h \geq 1$,
5. $G \cong \text{PSL}(2, p^h)$, $p$ being a prime, $h \geq 1$.

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