CHARACTERIZATION OF $A_5$ AND $PSL(2,7)$ BY SUM OF ELEMENT ORDERS

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Communicated by Bijan Taeri

Abstract. Let $G$ be a finite group. We denote by $\psi(G)$ the integer $\sum_{g \in G} o(g)$, where $o(g)$ denotes the order of $g \in G$. Here we show that $\psi(A_5) < \psi(G)$ for every non-simple group $G$ of order 60, where $A_5$ is the alternating group of degree 5. Also we prove that $\psi(PSL(2,7)) < \psi(G)$ for all non-simple groups $G$ of order 168. These two results confirm the conjecture posed in [J. Algebra Appl., 10 No. 2 (2011) 187-190] for simple groups $A_5$ and $PSL(2,7)$.

1. Introduction

Let $G$ be a finite group. We define the function

$$\psi(G) = \sum_{g \in G} o(g),$$

where $o(g)$ denotes the order of $g \in G$. We propose the following general question:

**Question 1.1.** What information about a group $G$ can be obtained from $\psi(G)$ and $|G|$?

The starting point on studying the function $\psi$ is in [1], where the maximum value of $\psi$ on the groups of the same order is investigated. In fact it is proved that

**Theorem 1.2.** Let $C$ be a cyclic group of order $n$. Then $\psi(G) < \psi(C)$ for all non-cyclic groups of order $n$. 

MSC(2010): Primary: 20D60; Secondary: 20D06.

Keywords: Finite groups, simple group, element orders.

Received: 13 May 2012, Accepted: 17 October 2012.
It follows that the cyclic groups are determined by their orders and sum of element orders.

In general, the invariants $|G|$ and $\psi(G)$ do not determine $G$. For example, there are two non-isomorphic groups $G_1$ and $G_2$ of order 27 such that $\psi(G_1) = \psi(G_2)$.

Note that the function $\psi$ is multiplicative, that is if $G_1$ and $G_2$ are two groups satisfying $\gcd(|G_1|, |G_2|) = 1$, then $\psi(G_1 \times G_2) = \psi(G_1)\psi(G_2)$.

Following Theorem 1.1, one can ask about the structure of groups having the minimum sum of element orders on all groups of the same order. In [2] it is proved that:

**Theorem 1.3.** Let $G$ be a nilpotent group of order $n$. Then $\psi(G) \leq \psi(H)$ for every nilpotent group $H$ of order $n$ if and only if each Sylow subgroup of $G$ is of prime exponent.

**Theorem 1.4.** Let $n$ be a positive integer such that there exists a non-nilpotent group of order $n$. Then there exists a non-nilpotent group $K$ of order $n$ with the property that $\psi(K) < \psi(H)$ for every nilpotent group $H$ of order $n$.

In other words the minimum value of $\psi(G)$ occurs on a non-nilpotent group, where $G$ varies on all groups of the same order.

Also it is conjectured in [2] that:

**Conjecture 1.5.** Let $S$ be a simple group. If $G$ is a non-simple group of order $|S|$, then $\psi(S) < \psi(G)$.

In other words if $n$ is a natural number such that there is a simple group of order $n$, then the minimum of $\psi$, on all groups of order $n$, occurs in a simple group. Here we confirm Conjecture 1.5 for $A_5$, the alternating group of degree 5, and $PSL(2, 7)$, the projective special linear group of $2 \times 2$ matrices over the field of order 7. It is hoped that the methods be useful for some other simple groups. Note that we determine $A_5$ and $PSL(2, 7)$ by their orders and sum of element orders.

Most of our notations are standard. If $p$ is a prime, then $n_p = n_p(G)$ denotes the number of Sylow $p$-subgroups of $G$ and the set of all Sylow $p$-subgroups of $G$ is denoted by $Syl_p(G)$ . If $n$ is a positive integer, then $C_n$ is a cyclic group of order $n$.

2. **Minimum of $\psi$ on all groups of order 60**

It is well-known that $A_5$ has 15 elements of order 2, 20 elements of order 3 and 24 elements of order 5. Therefore $\psi(A_5) = 211$.

**Theorem 2.1.** Let $G$ be any group of order 60. Then $\psi(G) \geq 211$ and $\psi(G) = 211$ if and only if $G \cong A_5$.

**Proof.** Since the order of $G$ is 60, the number of Sylow 5-subgroups is 1 or 6 and $n_3 = 1$ or 10. If $n_3 = 1$ or $n_5 = 1$, then $G$ contains a cyclic subgroup of order 15. Thus $G$ contains at least 8 elements of order 15. Also $G$ contains at least four elements of order 5 and at least two elements of order 3. So $G$ contains at most 45 elements of order at least 2. Hence

$$\psi(G) \geq 1 + 45(2) + 2(3) + 4(5) + 8(15) = 236.$$
So we may assume that \( n_3 = 10 \) and \( n_5 = 6 \). Then \( G \) contains 20 elements of order 3 and 24 elements of order 5. If \( I = \{ x \in G | o(x) = 3 \text{ or } 5 \} \), then \( |I| = 44 \). Therefore if there exits an element of order greater than 2 in \( G \setminus I \), then \( \psi(G) > 211 \). So suppose that each non-identity element of \( G \setminus I \) has order 2 which implies that \( |C_G(x)| = 4 \) for every element of order 2. It yields that the intersection of two distinct Sylow 2-subgroups of \( G \) is trivial and so \( n_2 = 5 \). Hence \( G \) is isomorphic to a subgroup of \( S_5 \), the symmetric group on 5 letters. It follows that \( G \cong A_5 \). This completes the proof.

\[ \square \]

**Corollary 2.2.** If \( G \) is a non-simple group of order 60, then \( \psi(G) > \psi(A_5) \).

3. **Minimum of \( \psi \) on all groups of order 168**

It is easy to check that \( PSL(2, 7) \) has 21 elements of order 2 and 42 elements of order 4, since \( PSL(2, 7) \) has 21 Sylow 2-subgroups isomorphic to \( D_8 \). Also \( PSL(2, 7) \) contains 56 elements of order 3 and 48 elements of order 7 because \( n_3 = 28 \) and \( n_7 = 8 \). Therefore \( \psi(PSL(2, 7)) = 715 \).

**Lemma 3.1.** Let \( G \) be a group of order 168. If \( G \) contains an element of order 21, then \( \psi(G) > 715 \).

**Proof.** Suppose first that \( G \) contains at least two cyclic subgroups of order 21. Then \( G \) has at least \( 2\phi(21) = 24 \) elements of order 21 and so \( G \) contains at most 144 elements which are not of order 21. Therefore \( \psi(G) > 24(21) + 143(2) + 1 = 791 > 715 \).

Now suppose that \( G \) contains unique cyclic subgroup \( T \) of order 21. Then \( n_7(G) = 1 \) and \( n_3(G) = 1 \). It follows that

\[
\frac{G}{C_G(P)} \rightarrow Aut(C_7),
\]

where \( P \) is the Sylow 7-subgroup of \( G \). Therefore \( |C_G(P)| = 2^3 \cdot 3 \cdot 7 \text{ or } 2^2 \cdot 3 \cdot 7 \). In the former case \( P \) is a central subgroup of \( G \). It follows from [1, Corollary B] that \( \psi(G) = \psi(\frac{G}{P})\psi(P) > 715 \). If \( |C_G(P)| = 2^2 \cdot 3 \cdot 7 \), \( P \) is central in \( C_G(P) \) and so \( \psi(C_G(P)) = 43(24) > 715 \), as desired.

\[ \square \]

**Lemma 3.2.** Let \( G \) be a group of order 168. If \( G \) contains no element of order 21 and \( n_7 = 8 \), then either \( \psi(G) > 715 \) or \( G \cong PSL(2, 7) \).

**Proof.** By hypothesis, \( K = N_G(P) = QP \), where \( P \in Syl_7(G) \) and \( Q \in Syl_3(G) \). Since \( n_3(K) = 7 \), \( n_3(G) \geq 7 \) and so \( n_3 = 7 \) or 28.

If the intersection of two conjugates \( K_1 \) and \( K_2 \) of \( K \) is trivial, then \( |K_1K_2| > 168 \), a contradiction. Therefore the intersection of any two conjugates of \( K \) has order 3 and since \( K \) has eight conjugates, \( n_3 = 28 \). This implies that \( N_G(Q) \cong S_3 \) or \( C_6 \).

If \( N_G(Q) \cong C_6 \), then \( G \) contains 56 elements of order 6 and so \( \psi(G) > 48(7) + 56(3) + 56(6) + 1 > 715 \).

If \( N_G(Q) \cong S_3 \), then \( C_G(Q) = Q \) and so the centralizer subgroup of any \( p \)-element is a \( p \)-subgroup for any prime \( p \). If \( n_2 \leq 7 \), then \( G \) contains at most 49 non-identity 2-elements. Since \( |G| = 168 \), there exists an element of \( G \) which is not a \( p \)-element, a contradiction. Hence \( n_2 = 21 \). If the intersection of any two distinct Sylow 2-subgroups is trivial, then \( |G| > 168 \). Therefore there exists \( x \in T_1 \cap T_2 \),
where $T_i \in \text{Syl}_2(G)$ for $i = 1, 2$. If $T_1$ is abelian then $|C_G(x)| > 8$, which is a contradiction. So $T_1$ is not abelian. It follows that $T_1 \cong D_8$ or $Q_8$. If $o(x) = 4$, then $C_G(x) = \langle x \rangle$ and so $x$ has 42 conjugates in $G$. Now $G$ has 48 elements of order 7, 56 elements of order 3 and 42 elements of order 4. This implies that $G$ must have 21 elements of order 2. Thus in the latter case $G$ has no non-trivial minimal normal subgroup and so $G$ is simple. Therefore $G \cong \text{PSL}(2, 7)$. This completes the proof. \hfill $\square$

**Lemma 3.3.** Let $G$ be a group of order 168. If $G$ contains no element of order 21 and $n_7 = 1$, then $\psi(G) > 715$.

**Proof.** Suppose that $P$ is the unique Sylow 7-subgroup of $G$. Then $|C_G(P)| = 2^3 \cdot 7$ or $2^2 \cdot 7$.

If $|C_G(P)| = 2^3 \cdot 7$, then $C_G(P) = P \times T$, where $T \in \text{Syl}_2(G)$. It follows that $n_2(G) = 1$, since $C_G(P)$ is normal in $G$. It yields that $\psi(C_G(P)) \geq 5 \cdot 43 = 215$. Since the order of each element in $G \setminus C_G(P)$ is at least 3 and $|G \setminus C_G(P)| = 112$, we have $\psi(G) \geq 215 + 112(3) > 715$.

Now suppose that $|C_G(P)| = 2^2 \cdot 7$. Then $C_G(P) = D \times P$, where $|D| = 4$. Since $C_G(P)$ is normal in $G$ and $D$ is characteristic in $C_G(P)$, $D$ is normal in $G$. So $D$ is the intersection of all Sylow 2-subgroups of $G$.

This is clear that $n_3 = 7$ or 28 and $n_2 = 7$ or 21. Set

$$E = \{x \in G | x \text{ is a non-identity 2-element} \},$$

and

$$S = \{x \in G | x \text{ is a non-identity 3-element} \}.$$ 

It follows that $|E| = 31$ or 87 and $|S| = 14$ or 56. Note that if $x \in G \setminus (E \cup S \cup P)$, then either $o(x) = 6$ or $o(x) \geq 12$. Also if $o(x) = 6$, then $G$ contains $2n_3(G)$ elements of order 6.

If $D \cong C_4$, then $\psi(C_G(P)) = 473$. Since $|G \setminus C_G(P)| = 140$, $\psi(G) \geq 473 + 140(2) > 715$.

If $D \cong C_2 \times C_2$, then $G$ has 18 elements of order 14. Now we consider four following cases:

1- If $|E| = 31$ and $|S| = 14$, then

$$\psi(G) \geq 31(2) + 14(3) + 6(7) + |G \setminus (E \cup S \cup P)|6 > 715.$$

2- If $|E| = 87$ and $|S| = 14$, then

$$\psi(G) \geq 87(2) + 14(3) + 6(7) + |G \setminus (E \cup S \cup P)|6 > 715.$$

3- If $|E| = 31$ and $|S| = 56$, then

$$\psi(G) \geq 31(2) + 56(3) + 6(7) + |G \setminus (E \cup S \cup P)|6 > 715.$$

4- If $|E| = 87$ and $|S| = 56$, then $G$ has 21 Sylow 2-subgroups. Suppose that $T \in \text{Syl}_2(G)$ and $T \cong C_2 \times C_2 \times C_2$. If $x \in D$, then $C_G(x) = TP$, since $|E \cup S \cup P| = 150$ and $G$ has 18 elements of order 14. Therefore $n_2(C_G(x)) = 7$. This is a contradiction because $C_G(x)$ contains all 21 Sylow 2-subgroups of $G$. Hence $T$ is not isomorphic to $(C_2)^3$. Thus each Sylow 2-subgroup of $G$ contains at least one cyclic subgroup of order 4. Since $D$ is isomorphic to $C_2 \times C_2$, the intersection of any two
distinct Sylow 2-subgroups of $G$ does not have any element of order 4. It follows that $G$ contains $21$ cyclic subgroups of order 4. Thus

$$\psi(G) \geq 87(2) + 56(3) + 18(14) + 42(4) > 715.$$ 

This completes the proof. $\Box$

**Theorem 3.4.** Let $G$ be any group of order 168. Then $\psi(G) \geq 715$.

**Proof.** It follows from Lemmas 3.1, 3.2 and 3.3. $\Box$

**Corollary 3.5.** Let $G$ be any non-simple group of order 168. Then $\psi(G) > \psi(PSL(2,7))$.

**Proof.** If $G$ satisfies the hypothesis of Lemmas 3.1 or 3.3, then the result holds. If $G$ satisfies the hypothesis of Lemma 3.2, then $\psi(G) > 715$, since $G$ is not simple. $\Box$

**References**


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