ON THE NUMBER OF THE IRREDUCIBLE CHARACTERS OF FACTOR GROUPS

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Dedicated to Professor Hossein Doostie on the occasion of his retirement

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Abstract. Let $G$ be a finite group and let $N$ be a normal subgroup of $G$. Suppose that $\text{Irr}(G|N)$ is the set of the irreducible characters of $G$ that contain $N$ in their kernels. In this paper, we classify solvable groups $G$ in which the set $C(G) = \{\text{Irr}(G|N)|1 \neq N \leq G\}$ has at most three elements. We also compute the set $C(G)$ for such groups.

1. Introduction

Let $G$ be a finite group and let $N$ be a normal subgroup of $G$. Suppose that $\text{Irr}(G|N)$ is the set of the irreducible characters of $G$ that contain $N$ in their kernels. Our aim in this paper is to study the set $C(G) = \{\text{Irr}(G|N)|1 \neq N \leq G\}$. Indeed, we classify finite solvable groups $G$ in which the set $C(G)$ has at most three elements and compute the set $C(G)$ for these groups. We are motivated by the article [4], where the author and S. Zandi considered a similar problem for conjugacy classes of $G$. They defined $\xi(N)$ to be the number of the conjugacy classes of $G$, contained in the normal subgroup $N$ and classified finite solvable groups $G$ in which the set $\mathcal{K}(G) = \{\xi(N)|N \leq G, N \neq G\}$ contains at most three element. It is easy to see that $|\mathcal{K}(G)| = 1$ if and only if $|C(G)| = 1$. This is equivalent to the simplicity of the group $G$. It is also routine to check that for solvable groups $G$, $|C(G)| = 2$ if and only if $|\mathcal{K}(G)| = 2$. However, we give examples of solvable groups $G$ with $|\mathcal{K}(G)| \neq |C(G)|$. In this paper we only consider finite solvable groups. An elementary abelian $p$-group of order $p^n$ is denoted by $C_p^n$. By a Frobenius group $G$ of type $C_p^n \rtimes C_q^m$, we mean that $G$ has an elementary abelian


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kernel of order $p^n$ and a cyclic complement of order $q^m$, where $p^n, q^m$ are prime powers. The set of the irreducible characters of $G$ is denoted by $\text{Irr}(G)$. Recall that for a normal subgroup $N$ of $G$, there exists a one to one correspondence between $\text{Irr}(G|N)$ and $\text{Irr}(G/N)$. We write $G = K \rtimes H$ if $K$ is an elementary abelian $p$-group and the action of $H$ on $K$ is non-trivial and irreducible. A monolith is a group with a unique minimal normal subgroup. Our notations are standard and mainly obtained from [3]. The main result of this paper is the following:

**Theorem A.** Let $G$ be a solvable group. If $|\mathcal{C}(G)| = 3$, then one of the following holds ($p, q$ and $r$ are prime numbers and $p \neq q$):

(i) $G$ is an abelian group of order $pq$ and $\mathcal{C}(G) = \{1, p, q\}$.

(ii) $G$ is a group of order $p^3$ and $\mathcal{C}(G) = \{1, p, p^2\}$.

(iii) $G$ is a Frobenius group of type $\mathbb{C}_p^3 \rtimes \text{Irr} \mathbb{C}_q^2$ and $\mathcal{C}(G) = \{1, q, q^2\}$.

(iv) $G = K \rtimes H$, where $K \cong \mathbb{C}_p^3$ and $\mathbb{C}_q^2$, where $p^3 - 1 = q^2(q - 1)$ and the action is non-faithful. Also, $\mathcal{C}(G) = \{1, q, q^2\}$.

(v) $G$ is a Frobenius group of order $\mathbb{C}_p^{2n} \rtimes \mathbb{C}_q$ and contains two minimal normal subgroups of order $p^n$. Also, we have $\mathcal{C}(G) = \{1, q, q + (p^n - 1)/q\}$.

(vi) $G$ has exactly two non-trivial proper normal subgroups, namely $N$ and $G'$, where $N < G'$. Also, $\mathcal{C}(G) = \{1, q, q + (|G' : N| - 1)/q\}$, where $q = |G : G'|$.

**Remark 1.1.** By the results of [4], a group $G$ satisfies the statement (vi) if and only if $G$ is one of the following groups:

- $G = K \rtimes H$, $K$ is a $p$-group which is either special or abelian, $H \cong \mathbb{C}_q$ and $\Phi(K), K$ are the only non-trivial proper normal subgroups of $G$.
- $G = L \rtimes H$, where $L = G''$ and $H$ is a Frobenius group of type $\mathbb{C}_p^3 \rtimes \text{Irr} \mathbb{C}_q$.

Comparing Theorem A with the main theorem of [4], we get the following result.

**Corollary B.** Let $G$ be a solvable group and let $\mathcal{C}(G) \leq 3$. Then $|\mathcal{C}(G)| = |\mathcal{K}(G)|$, unless $G$ is not a monolith and $G'$ is a minimal normal subgroup of $G$.

We will give examples of solvable groups in which $|\mathcal{C}(G)| \neq |\mathcal{K}(G)|$. According to Corollary B, our examples are of type (iv).

2. Preliminaries

We start this section with some easy results.

**Lemma 2.1.** Let $G$ be a group with $|\mathcal{C}(G)| = t$. Assume that $N_1, ..., N_t$ are non-trivial normal subgroups of $G$. If $N_1 \leq ... \leq N_t$, then $N_t = G$. 
Lemma 2.2. Let $G$ be an abelian group of order $n$. Then $C(G) = D(n) - \{n\}$, where $D(n)$ is the set of the positive integers dividing $n$.

Lemma 2.3. Let $G$ be a group and let $N$ be a subgroup of $G$. If $G' \leq N$, then $|\text{Irr}(G|N)| = |G : N|$. In particular, if $N_1, N_2$ contain $G'$, then $|\text{Irr}(G|N_1)| = |\text{Irr}(G|N_2)|$ if and only if $|N_1| = |N_2|$.

Lemma 2.4. \[3\] Theorem 5.6] Let $G$ be a group with an abelian Sylow $p$-subgroup. Then $G' \cap Z(G)$ is a $p'$-group.

Lemma 2.5. \[3\] Lemma 12.11] Let $G$ be a non-abelian group. Then $\text{cd}(G) = \{1, p\}$, where $p$ is a prime if and only if one of the followings hold:

2. $|G : Z(G)| = p^3$.

Lemma 2.6. \[4\] Lemma 2.6] Let $G$ be a solvable group and assume that $N$ is a proper normal subgroup of $G$. Then $G' \cap N \leq G$. In particular if $G'$ is a maximal subgroup of $G$, then it contains all normal subgroups of $G$.

Lemma 2.7. \[4\] Lemma 2.8] Let $G$ be a group and $G = A \rtimes H$, where $A$ is an abelian normal subgroup of $G$ and $H \cong \mathbb{C}_p$ for a prime $p$. If $Z(G) = 1$, then $G$ is a Frobenius group with kernel $A$.

Lemma 2.8. \[3\] Lemma 12.3] Let $G$ be a solvable group. If $G'$ is the unique minimal normal subgroup of $G$, then one of the followings holds:

1. $G$ is a $p$-group, $|G'| = p$ and $Z(G)$ is cyclic.
2. $G$ is a Frobenius group of type $C_p \rtimes \text{Irr} \mathbb{C}_{q^n}$.

Theorem 2.9. \[1\] 10.4] (Gaschütz’s Theorem) Let $G$ be a group and assume that $P$ is a Sylow $p$-subgroup of $G$. If $K$ is an abelian subgroup of $P$ and $K \leq G$, then $G$ splits over $K$ if and only if $P$ splits over $K$.

Theorem 2.10. \[1\] 18.1] (Schur-Zassenhaus Theorem) Let $G$ be a group and assume that $H$ is a Hall normal subgroup of $G$. Then $G$ splits over $H$.

3. Main results

In this section, we prove Theorem A. First, we prove the case when $G$ is nilpotent.

Proposition 3.1. Let $G$ be a nilpotent group with $|C(G)| = 3$. Then one of the followings hold:

1. $G$ is an abelian group of order $pq$, where $p, q$ are distinct prime numbers. Also, $C(G) = \{1, p, q\}$.
2. $G$ is a group of order $p^3$ and $C(G) = \{1, p, p^2\}$.

Proof. If $G$ is not a $p$-group, then we may choose maximal subgroups $M, L$ of $G$ such that $|G : M| \neq |G : L|$. Then, $G' \leq L \cap M$ and we deduce by Lemma 2.3 that $|\text{Irr}(G|L)| \neq |\text{Irr}(G|M)|$. Therefore, $G' = 1$ and $G$ is abelian. Consequently, Lemma 2.2 implies that $|G| = pq$ and $C(G) = \{1, p, q\}$. Next,
assume that $G$ is a $p$-group. Thus by Lemma 2.1 $G$ must be of order $p^3$. If $G$ is abelian, then it is clear that $C(G) = \{1, p, p^2\}$. Now, suppose that $G$ is not abelian. Then, $|G'| = p$ and all non-trivial normal subgroups of $G$ contain $G'$. Hence, we deduce by Lemma 2.3 that $C(G) = \{1, p, p^2\}$. The proof is now completed.

We are now ready to prove the main result of this paper. During the proof, we will frequently use the following well-known equality:

$$|G| = |G : G'| + \sum_{\chi \in \text{Irr}_1(G)} \chi(1)^2.$$  

(3.1)

Here $\text{Irr}_1(G)$ is the set of nonlinear irreducible characters of $G$.

**Proof of Theorem A.** If $G$ is nilpotent, then by Proposition 3.1 $G$ is in case (i) or (ii). So, we may assume that $G$ is not nilpotent. We finish the proof through the following steps:

**Step 1.** If $G$ is a monolith, then $G$ is either in case (iii) or (vi).

Let $N$ be the unique minimal normal subgroup of $G$. If $N < G'$, then $G'$ is a maximal subgroup of $G$ and by Lemma 2.6 it contains all proper normal subgroup of $G$. Therefore by Lemma 2.1 $N$ and $G'$ are the only non-trivial proper normal subgroups of $G$. Hence, $G$ is of type (vi). Note that $|G : G'| = q$ is a prime number. Also, $G'/N$ is an abelian maximal subgroup of $G/N$. So by Lemma 2.5 $\text{cd}(G/N) = \{1, q\}$ and we may write:

$$G/N = |G : G'| + (|\text{Irr}(G|N)| - |G : G'|)q^2.$$  

(3.2)

This implies that $|\text{Irr}(G|N)| = q + (|G' : N| - 1)/q$. Next, assume that $G'$ is the unique minimal normal subgroup of $G$. Then, Lemma 2.8 implies that $G$ is a Frobenius group of type $\mathbb{C}_p \rtimes \text{Irr} \mathbb{C}_{q^m}$. Note that $|G : G'| = q^m$ and $G$ has normal subgroups of index $q^i$ for $i = 0, 1, ..., m$. Since $|C(G)| = 3$, we conclude by Lemma 2.1 that $m = 2$. That is, $G$ is of type (iii). Now, choose a proper normal subgroup of $N$ of $G$ with $G' < N$. Then, $N$ is of index $q$. On the other hand, since the action is irreducible and $Z(G) = 1$, we conclude that $G'$ is the unique minimal normal subgroup of $G$. So, all non-trivial proper normal subgroups of $G$ have index $q$. Hence, $C(G) = \{1, q, q^2\}$.

**Step 2.** If $G$ is not a monolith and $G'$ is a minimal normal subgroup, then $G$ is in case (iv).

Let $N$ be a minimal normal subgroup of $G$ and $N \neq G'$. Then, $N$ is central. We claim that $N = Z(G)$, otherwise, considering the subgroups $G'$, $N$, $Z(G)$ and $G'Z(G)$, we may find at least four elements in $C(G)$ which is a contradiction. So, $N = Z(G)$ and $G$ has no other minimal normal subgroups. Let $|Z(G)| = q$ and $|G'| = p^n$. If $r$ is a prime divisor of $G$ apart from $p, q$, then $G$ contains normal subgroups of order $rp^n$ and $qp^n$, containing $G'$. Therefore, $|C(G)| \geq 4$, a contradiction. So, $p, q$ are the
only prime divisors of $|G|$. Now, Lemma 2.5 implies that, $\text{cd}(G) = \{1, q\}$ where $q = |G : G'Z(G)|$. Applying the equality (3.1) to $G/Z(G)$ we have:

$$qp^n = q + q^2 (|\text{Irr}(G|N)| - q)$$
$$\Rightarrow qp^n = q + q^2 (q^2 - q)$$
$$\Rightarrow q^2 (q - 1) = p^n - 1$$

(3.3)

On the other hand, by Theorem 2.10 $G$ splits over $G'$ and we have $G \cong G' \rtimes G/G'$. Therefore, $G$ is in case (iv). Now assume that $G$ is of type (iv). We show that $C(G) = \{1, q, q^2\}$. It is easy to see that $K = Z(G)$ and $G' = H'$ and $G'Z(G)$ are the only non-trivial proper normal subgroups of $G$. The equality (3.3) garantees that $|\text{Irr}(G|Z(G))| = |\text{Irr}(G|G'))|$. Therefore, $C(G) = \{1, q, q^2\}$. This completes the proof of this step.

Step 3. If $G$ is not a monolith and $G'$ is not a minimal normal subgroup, then $G$ is in case (v).

Let $N, L$ be distinct minimal normal subgroups of $G$. Then $NL = G'$ and $G'$ is a maximal subgroup of index $q$. Since $G'/L$ and $G'/N$ are abelian, we conclude by Lemma 2.5 that $\text{cd}(G/L) = \text{cd}(G/N) = \{1, q\}$. Also note that $|\text{Irr}(G|N)| = |\text{Irr}(G|L)|$. Thus, applying the equality (3.1) to the groups $G/L$ and $G/N$, we conclude that $|N| = |L|$. Hence, $G'$ is a $p$-group and we deduce by Theorem 2.10 that $G \cong \mathbb{C}_p \rtimes \mathbb{C}_q$. Now by Lemma 2.4 and Lemma 2.7 we get $G$ is in case (v). Next, we show that $C(G) = \{1, q, q + (p^n - 1)/q\}$. Let $N$ be an arbitrary minimal normal subgroup $G$ and $N \neq G'$. As $Z(G) = 1$, we must have $N < G'$. So, $G'$ contains all minimal normal subgroups of $G$. This implies that $|G'| = p^{2n}$ and that $G$ has exactly two minimal normal subgroups. Certainly, $|\text{Irr}(G|G'))| = q$. Let $N$ be a minimal normal subgroup. Since $G'/N$ is an abelian maximal subgroup, then the equality (3.2) is valid. Therefore, $|\text{Irr}(G|N)| = q + (|G'| : N - 1)/q$. As $|G'| : N = p^n$, the result follows. □

Example 3.2. Consider the group $G = \text{SmallGroup}(20, 1)$, the first group of order 20 in the library of GAP [2]. Then, it is easy to see that $G$ is case (iv) in Theorem A. Therefore, $C(G) = \{1, 2, 4\}$. However, we may check that $K(G) = \{1, 2, 5, 6\}$. Now, assume that $H = \text{SmallGroup}(63, 1)$. Then, $K(G) = \{1, 3, 9\}$, while $C(G) = \{1, 3, 5, 9\}$. In both examples, as Corollary B implies, the groups contain more than one minimal normal subgroup, one of which is the derived subgroup.

References

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