SOME RESULTS ON CHARACTERIZATION OF FINITE GROUPS BY NON-COMMUTING GRAPH

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Abstract. The non commuting graph $\nabla(G)$ of a non-abelian finite group $G$ is defined as follows: its vertex set is $G - Z(G)$ and two distinct vertices $x$ and $y$ are joined by an edge if and only if the commutator of $x$ and $y$ is not the identity. In this paper we prove some new results about this graph. In particular we will give a new proof of Theorem 3.24 of [A. Abdollahi, S. Akbari, H. R, Maimani, Non-commuting graph of a group, J. Algebra, 298 (2006) 468-492]. We also prove that if $G_1, G_2, \ldots, G_n$ are finite groups such that $Z(G_i) = 1$ for $i = 1, 2, \ldots, n$ and they are characterizable by non commuting graph, then $G_1 \times G_2 \times \cdots \times G_n$ is characterizable by non-commuting graph.

1. Introduction

Let $G$ be a finite group. The non-commuting graph $\nabla(G)$ of $G$ is defined as follows: the set of vertices of $\nabla(G)$ is $G - Z(G)$, where $Z(G)$ is the center of $G$ and two vertices $x$ and $y$ are connected whenever $[x, y] \neq 1$, where $[x, y]$ is the commutator of $x$ and $y$. In [1] the authors put forward a conjecture as follows:

Conjecture 1. Let $G$ be a finite non-abelian nilpotent group and $H$ be a group such that $\nabla(G) \cong \nabla(H)$. Then $H$ is nilpotent.

In this paper we prove this conjecture in the case of $|G| = |H|$. In fact this is proved in [1], but our proof is different. We say $G$ is factorizable if $G$ is isomorphic to a direct product of its proper subgroups. We will show that if $G$ and $H$ are two centerless groups and $\nabla(G) \cong \nabla(H)$, then $G$ is factorizable if and only if $H$ is factorizable. Moreover if $G \cong G_1 \times G_2 \times \cdots \times G_n$, then there are

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subgroups of \( H \) say \( H_1, H_2, \ldots, H_n \) such that \( H \cong H_1 \times H_2 \times \cdots \times H_n \). \( G \) is called characterizable by non-commuting graph if, when \( H \) is an arbitrary group with \( \nabla(G) \cong \nabla(H) \), then \( G \cong H \). We prove that if \( G_1, G_2, \ldots, G_n \) are finite groups such that \( Z(G_i) = 1 \) for \( i = 1, 2, \ldots, n \) and \( G_i \) is characterizable by non-commuting graph, then \( G_1 \times G_2 \times \cdots \times G_n \) is characterizable by non-commuting graph. In [3] Ron Solomon and Andrew Woldar proved that all finite non-abelian simple groups are characterizable by non-commuting graph.

2. Preliminaries

**Lemma 2.1.** Let \( G \) and \( H \) be two finite non-abelian groups. If \( \nabla(G) \cong \nabla(H) \), then \( \nabla(C_G(A)) \cong \nabla(C_H(\varphi(A))) \) for all \( \varphi \) is the isomorphism from \( \nabla(G) \) to \( \nabla(H) \) and \( C_G(A) \) is non-abelian.

**Proof.** It is sufficient to show that \( \varphi |_{V(C_G(A))}: V(C_G(A)) \rightarrow V(C_H(\varphi(A))) \) is onto, where \( \varphi |_{V(C_G(A))} \) is the restriction of \( \varphi \) to \( V(C_G(A)) \) and

\[
\begin{align*}
V(C_G(A)) &:= C_G(A) - Z(C_G(A)), \\
V(C_H(\varphi(A))) &:= C_H(\varphi(A)) - Z(C_H(\varphi(A)))
\end{align*}
\]

Assume that \( d \) is an element of \( V(C_H(\varphi(A))) \). Then \( d \in H - Z(H) \) and so there exists an element \( c \) of \( G - Z(G) \) such that \( \varphi(c) = d \). From \( d = \varphi(c) \in C_H(\varphi(A)) \), it follows that \( [\varphi(c), \varphi(g)] = 1 \) for all \( g \in A \) and since \( \varphi \) is an isomorphism from \( \nabla(G) \) to \( \nabla(H) \), \( [c, g] = 1 \) for all \( g \in A \). Therefore \( c \in C_G(A) \). But \( d \notin Z(C_H(\varphi(A))) \), so for an element \( x \in C_H(\varphi(A)) \) we have \( [x, d] \neq 1 \). Hence \( x \) is an element of \( H \) that does not commute with \( d \in H \). This implies that \( x \in H - Z(H) \). Thus there exists \( x' \in G - Z(G) \), such that \( \varphi(x') = x \). It is easy to see that \( [x', c] \neq 1 \) and therefore \( c \notin Z(C_G(A)) \). Therefore \( c \in C_G(A) - Z(C_G(A)) = V(C_G(A)) \). Hence \( \varphi(c) = d \).

We denote by \( I_G \) the set of all bijections \( \phi: G \rightarrow G \) such that \( [x, y] = 1 \) if and only if \( [\phi(x), \phi(y)] = 1 \) for all \( x, y \in G \). It is easy to see that \( I_G \) is a subgroup of \( S_G \), where \( S_G \) is the symmetric group on \( G \).

**Lemma 2.2.** Let \( G \) be a finite non-abelian group. Then \( \text{Aut}(G) \leq I_G \), where \( \text{Aut}(G) \) is the automorphism group of \( G \).

**Proof.** Suppose that \( \psi \in \text{Aut}(G) \). If \( x, y \in G \) are two arbitrary elements of \( G \), then \( [x, y] = 1 \) if and only if \( (x, y)\psi = 1 \) and \( [x\psi, y\psi] = 1 \) and the proof is complete.

**Lemma 2.3.** Let \( G \) and \( H \) be two finite non-abelian groups with \( \nabla(G) \cong \nabla(H) \) and \( |G| = |H| \). Then \( I_G \cong I_H \).

**Proof.** Since \( \nabla(G) \cong \nabla(H) \), \( |G - Z(G)| = |H - Z(H)| \). But \( |G| = |H| \) and so \( |Z(G)| = |Z(H)| \). Thus there is a bijection \( \alpha \) from \( Z(G) \) to \( Z(H) \). Moreover since \( \nabla(G) \cong \nabla(H) \), there is a graph isomorphism
\( \varphi \) from \( G - Z(G) \) to \( H - Z(H) \). We define \( \psi : I_G \to I_H \) by

\[
\psi(\delta)(x) = \varphi \circ \delta \mid_{G - Z(G)} \circ \varphi^{-1}(x)
\]

if \( x \notin Z(H) \) and

\[
\psi(\delta)(x) = \alpha \circ \delta \mid_{Z(G)} \circ \alpha^{-1}(x)
\]

if \( x \in Z(H) \), for all \( \delta \in I_G \), where \( \circ \) denote the composition of functions. Routine checking shows that \( \psi \) is an isomorphism from \( I_G \) to \( I_H \) and so \( I_G \cong I_H \).

\[ \square \]

3. Results and Properties

**Proposition 3.1.** Let \( G \) be a finite non-abelian nilpotent group and \( H \) be a group such that \( \nabla(G) \cong \nabla(H) \) and \( |G| = |H| \). Then \( H \) is nilpotent.

**Proof.** We use induction on \( |G| = n \). Clearly if \( |G| = 1 \), then the assertion holds. Suppose the result is valid for all groups \( K \), with \( |K| < n \). We will prove Proposition 3.1 when \( |G| = n \). Since \( G \) is nilpotent, we can write \( G \cong P_1 \times P_2 \times \cdots \times P_k \), where \( P_i \) is the \( p_i \)-Sylow subgroup of \( G \) say of order \( p_i^{a_i} \) for \( i = 1, 2, \ldots, k \).

If \( G \) is a \( p \)-group for some prime number \( p \), then since \( |G| = |H| \), \( H \) is a \( p \)-group too and so \( H \) is nilpotent. If \( G = P \times A \), where \( P \) is a \( p \)-group and \( A \) is an abelian group, then \( \frac{G}{Z(G)} \) is a \( p \)-group and since \( |G| = |H| \) and \( |Z(G)| = |Z(H)| \), we conclude that \( \frac{H}{Z(H)} \) is a \( p \)-group and so \( H \) is nilpotent in this case.

Let \( \varphi \) be an isomorphism from \( \nabla(G) \) to \( \nabla(H) \). We extend \( \varphi \) to \( H \) by defining \( \varphi(z) = \psi(z) \), where \( \psi \) is an arbitrary bijective map from \( Z(G) \) to \( Z(H) \).

By above argument we may assume that \( k > 1 \) and \( G \) is not product of a \( p \)-group and an abelian group.

If \( C_G(P_i) = G \), for all \( i = 1, 2, \ldots, k \), then \( P_i \leq Z(G) \) for \( i = 1, 2, \ldots, k \) and so \( G = Z(G) \), a contradiction. Hence there is a Sylow-subgroup \( P_i \) of \( G \) such that \( C_G(P_i) \neq G \). But \( C_G(P_i) \) is nilpotent and \( \nabla(C_G(P_i)) \cong \nabla(C_H(\varphi(P_i))) \) by Lemma 2.1, where \( \varphi \) is an isomorphism from \( \nabla(G) \) to \( \nabla(H) \) and so \( C_H(\varphi(P_i)) \) is nilpotent by inductive hypothesis. Without loss of generality we assume that

\[
G = P_1 \times P_2 \times \cdots \times P_k, k > 1
\]

Let

\[
K = C_G(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)
\]

Thus

\[
K = Z(P_1) \times \cdots \times Z(P_{i-1}) \times P_i \times Z(P_{i+1}) \times \cdots \times Z(P_k)
\]

Therefore \( \frac{K}{Z(G)} \) is a \( p_i \)-group and so

\[
\frac{C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))}{Z(H)}
\]
is a $p_i$-group too, because $|K| = |C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))|$. On the other hand

$$Z(G) = Z(C_G(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))$$

This implies that

$$Z(H) = Z(C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))),$$

because $\varphi$ is an isomorphism from $\nabla(G)$ to $\nabla(H)$. Thus

$$\frac{C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))}{Z(C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)))}$$

is a nilpotent group and so $C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))$ is nilpotent. Moreover since

$$C_G(P_i) = P_1 \times \cdots \times P_{i-1} \times Z(P_i) \times P_{i+1} \times \cdots \times P_k,$$

we have $p_1^{a_1} \cdots p_{i-1}^{a_{i-1}} p_{i+1}^{a_{i+1}} \cdots p_k^{a_k} \mid |C_G(P_i)|$. Now if

$$p_1^{a_1} \cdots p_{i-1}^{a_{i-1}} p_{i+1}^{a_{i+1}} \cdots p_k^{a_k} \mid |C_G(A)|$$

for an arbitrary subset $A$ of $G$, then we have

$$P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k \leq C_G(A)$$

and since $Z(G) \leq C_G(A)$, we conclude that

$$P_1 \times \cdots \times P_{i-1} \times Z(P_i) \times P_{i+1} \times \cdots \times P_k = C_G(P_i) \leq C_G(A)$$

Therefore if $|C_G(A)| = |C_G(P_i)|$, then $C_G(A) = C_G(P_i)$ for all $A \subseteq G$. We know that

$$|C_H(\varphi(P_i))| = |C_H(h^{-1} \varphi(P_i)h)| = |h^{-1} C_H(\varphi(P_i))h|$$

for all $h \in H$. Thus

$$|C_G(P_i)| = |C_G(h^{-1} \varphi(P_i)h)|.$$  

Hence

$$C_G(P_i) = C_G(h^{-1} \varphi(P_i)h),$$

which implies that

$$C_H(\varphi(P_i)) = C_H(h^{-1} \varphi(P_i)h) = h^{-1} C_H(\varphi(P_i))h,$$

where $h$ is an arbitrary element of $H$. Therefore $C_H(\varphi(P_i)) \leq H$. By a similar argument we can see that

$$C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)) \leq H.$$

Obviously

$$\frac{|C_G(P_i)C_G(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)|}{|C_G(P_i)| |C_G(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)|} = [G].$$
Thus
\[ \frac{|C_H(\varphi(P_i))||C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n))|}{|C_H(\varphi(P_i)) \cap C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n))|} = |H| \]
and so
\[ C_H(\varphi(P_i))C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n)) = H \]
and since
\[ C_H(\varphi(P_i)) \quad \text{and} \quad C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n)) \]
are nilpotent normal subgroups of $H$, we conclude that $H$ is nilpotent.

\[ \square \]

**Proposition 3.2.** Let $G$ and $H$ be two finite non-abelian groups. If $\nabla(G) \cong \nabla(H)$ and $|Z(G)| = |Z(H)| = 1$, then $G$ is factorizable if and only if $H$ is factorizable. Moreover if $G \cong G_1 \times G_2 \times \cdots \times G_n$, then there are subgroups $H_1, H_2, \ldots, H_n$ of $H$ such that $H \cong H_1 \times H_2 \times \cdots \times H_n$ and $\nabla(G_i) \cong \nabla(H_i)$ for $i = 1, 2, \ldots, n$.

**Proof.** Without loss of generality assume that $G = G_1 \times G_2 \times \cdots \times G_n$. Put
\[ M_i = 1 \times \cdots \times G_i \times \cdots \times 1, \]
for $1 \leq i \leq n$. This implies that
\[ C_G(M_i) = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n, \]
for $1 \leq i \leq n$. Thus
\[ |C_G(M_i)||M_i| = |G|, \]
\[ C_G(C_G(M_i)) = M_i, \]
\[ M_i \cap C_G(M_i) = 1, \]
for $1 \leq i \leq n$. On the other hand
\[ C_G(M_1) \cap \ldots \cap C_G(M_{i-1}) \cap C_G(M_{i+1}) \cap \ldots \cap C_G(M_n) = M_i, \]
for $1 \leq i \leq n$. Therefore we have
\[ C_G(C_G(M_1) \cap \ldots \cap C_G(M_{i-1}) \cap C_G(M_{i+1}) \cap \ldots \cap C_G(M_n)) \cap C_G(C_G(M_i)) = 1, \]
for $1 \leq i \leq n$. Since $\nabla(G) \cong \nabla(H)$, there is an isomorphism, $\varphi$, from $\nabla(G)$ to $\nabla(H)$. Hence
\[ |C_H(\varphi(M_i))||\varphi(M_i)| = |H|, \]
\[ C_H(C_H(\varphi(M_i))) = \varphi(M_i) \]
and
\[ \varphi(M_i) \cap C_H(\varphi(M_i)) = 1, \]
\[ C_H(C_H(\varphi(M_1)) \cap \ldots \cap C_H(\varphi(M_{i-1})) \cap C_H(\varphi(M_{i+1})) \cap \ldots \cap C_H(\varphi(M_n))) \cap C_H(C_H(\varphi(M_i))) = 1, \]
for $i = 1, 2, \ldots, n$. Since
\[
\varphi(M_i) \cap C_H(\varphi(M_i)) = 1,
\]
\[
|\varphi(M_i)||C_H(\varphi(M_i))| = |H|
\]
and
\[
\varphi(M_i)C_H(\varphi(M_i)) = H,
\]
thus
\[
C_H(C_H(\varphi(M_i))) = \varphi(M_i) \trianglelefteq \varphi(M_i)C_H(\varphi(M_i)) = H.
\]
Therefore $\varphi(M_i)$ for $i = 1, 2, \ldots, n$ is a normal subgroup of $H$. Moreover we have
\[
\varphi(M_1) \ldots \varphi(M_{i-1})\varphi(M_{i+1}) \ldots \varphi(M_n) \subseteq C_H(C_H(\varphi(M_1)) \cap \ldots \cap C_H(\varphi(M_{i-1})) \cap C_H(\varphi(M_{i+1})) \cap \ldots \cap C_H(\varphi(M_n)))
\]
and so
\[
\varphi(M_1) \ldots \varphi(M_{i-1})\varphi(M_{i+1}) \ldots \varphi(M_n) \cap \varphi(M_i) \subseteq C_H(C_H(\varphi(M_1)) \cap \ldots \cap C_H(\varphi(M_{i-1})) \cap C_H(\varphi(M_{i+1})) \cap \ldots \cap C_H(\varphi(M_n))) \cap C_H(C_H(\varphi(M_i))) = 1,
\]
which implies that
\[
\varphi(M_1) \ldots \varphi(M_{i-1})\varphi(M_{i+1}) \ldots \varphi(M_n) \cap \varphi(M_i) = 1,
\]
for $i = 1, 2, \ldots, n$. Hence
\[
\varphi(M_1) \cap \varphi(M_2) \ldots \varphi(M_n) = 1,
\]
\[
\varphi(M_2) \cap \varphi(M_3) \ldots \varphi(M_n) = 1,
\]
\[
\ldots, \varphi(M_{n-1}) \cap \varphi(M_n) = 1.
\]
On the other hand $|M_1| \ldots |M_n| = |G|$. Now it is easy to see that
\[
\varphi(M_1)\varphi(M_2) \ldots \varphi(M_n) = H.
\]
Put $\varphi(M_i) = H_i$, $i = 1, 2, \ldots, n$. Therefore we have proved
\[
H \cong H_1 \times \cdots \times H_n.
\]
We know that
\[
G_i \cong M_i = C_G(C_G(M_i)).
\]
Thus $\nabla(G_i) \cong \nabla(C_G(C_G(\varphi(M_i))))$, because
\[
\nabla(C_G(C_G(M_i))) \cong \nabla(C_G(C_G(\varphi(M_i))))
\]
and since
\[ C_H(C_H(\varphi(M_i))) = \varphi(M_i) = H_i, \]
we conclude that \( \nabla(G_i) \cong \nabla(H_i) \) for \( i = 1, 2, \ldots, n \). \( \square \)

**Corollary 3.3.** Let \( G_1, G_2, \ldots, G_n \) be finite non-abelian groups. If \( G_1, G_2, \ldots, G_n \) are characterizable by non-commuting graph and \( Z(G_i) = 1 \) for \( i = 1, 2, \ldots, n \), then \( G_1 \times G_2 \times \cdots \times G_n \) is characterizable by non-commuting graph.

**Proof.** Assume that \( \nabla(H) \cong \nabla(G_1 \times G_2 \times \cdots \times G_n) \). Thus
\[
\nabla(C_H(\varphi(G_2 \times \cdots \times G_n))) \cong \nabla(C_{G_1 \times \cdots \times G_n}(G_2 \times \cdots \times G_n)) = \nabla(G_1).
\]
But \( G_1 \) is characterizable by non-commuting graph and so
\[ G_1 \cong C_H(\varphi(G_2 \times \cdots \times G_n)) \]
and since
\[ Z(C_{G_1 \times G_2 \times \cdots \times G_n}(G_2 \times \cdots \times G_n)) = Z(G_1) = 1, \]
we have
\[ Z(C_H(\varphi(G_2 \times \cdots \times G_n))) = 1. \]
It follows that \( Z(H) = 1 \). By Proposition 3.2, there are subgroups \( H_1, H_2, \ldots, H_n \) of \( H \) such that
\[ H \cong H_1 \times H_2 \times \cdots \times H_n \]
and \( \nabla(G_i) \cong \nabla(H_i) \) for \( i = 1, 2, \ldots, n \). But since \( G_i \) is characterizable by non-commuting graph, we have \( G_i \cong H_i, i = 1, 2, \ldots, n \) and so
\[ H_1 \times H_2 \times \cdots \times H_n \cong G_1 \times G_2 \times \cdots \times G_n. \]
Therefore \( H \cong G_1 \times G_2 \times \cdots \times G_n \). \( \square \)

**Corollary 3.4.** If \( S_1, S_2, \ldots, S_m \) are finite non-abelian simple groups, then \( S_1 \times S_2 \times \cdots \times S_m \) is characterizable by non-commuting graph.

**Proof.** In [3] the authors prove that all simple groups are characterizable by non-commuting graph. Thus by Corollary 3.3, direct product of simple groups are characterizable by non-commuting graph. \( \square \)

**Proposition 3.5.** Let \( G \) be a finite non-abelian group such that \( I_G = \text{Inn}(G) \) and \( Z(G) = 1 \), where \( \text{Inn}(G) \) is the group of inner automorphisms of \( G \). If \( H \) is a group with \( \nabla(G) \cong \nabla(H) \) and \( |G| = |H| \), then \( G \cong H \).
Proof. By Lemma 2.3 we have $I_G \cong I_H$. But $Z(G) = 1$, $Inn(G) \cong I_G$ and so we have $G \cong I_G$. Moreover $Z(H) = 1$ and by Lemma 2.2 we can write

$$H \cong Inn(H) \leq Aut(H) \leq I_H \cong I_G \cong G.$$ 

Therefore $H$ is embedded in $G$ and since $|H| = |G|$, we have $G \cong H$. 

□

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